2010 Part IA Formal Logic, Model Answers

1. Attempt all parts of this question

   (a) Carefully define the notions of

      (i) a truth-function

      A function is a map which assigns exactly one value to each given input. A truth-function is then a function whose values are truth values (True or False)

      (ii) a truth-functional connective

      A (sentential) connective is a way of forming complex sentences from one or more constituent sentences. A truth-functional connective is a connective where the truth-value of the complex sentence is completely determined by the truth-values of the constituent sentences (whatever they may be).

      It follows that a truth-functional connective expresses a truth-function, since it maps from one or more objects (the truth-values of the constituent sentences) to a single determinate truth-value (the truth-value of the complex sentence).

      (iii) an expressively adequate set of connectives

      A set of connectives, S is expressively adequate iff a language containing just those connectives is rich enough to express all (possible) truth-functions of the atomic sentences of the language. Otherwise put: S is an expressively adequate set of connectives iff given any arbitrary truth-table, it is possible to construct a formula with that truth-table using only the connectives in S.

      (iv) tautological entailment

      The wffs $A_1, \ldots, A_n$ tautologically entail the wff C if and only if there is no valuation of the atoms involved in the wffs which makes all of $A_1, \ldots, A_n$ true whilst simultaneously making C false.

   (b) Carefully explain the difference between what is symbolized by '$\supset$' and '$\vDash$'.

   The symbol '$\supset$' is the material conditional. This is a truth-functional connective of **PL**. Its truth-table is:

A	B	$A \supset B$
T	T	T
T	F	F
F	T	T
F	F	T

   When we attempt to regiment an English sentence like 'if…, then ____' in **PL**, our best effort is normally something of the form '$A \supset B$'.

   By contrast, the symbol '$\vDash$' is the double turnstile. It indicates tautological validity. To indicate that the wffs $A_1, \ldots, A_n$ tautologically entail C, we write '$A_1, \ldots, A_n \vDash C$'. Derivatively, we write '$\vDash C$' to indicate that C is a tautology, for this says that the **PL**-inference with no premises whose conclusion is C is tautologically valid. Likewise, we write '$C \vDash$' to indicate that 'C' is a contradiction.

   Crucially, then: '$\supset$' is a two-place truth-functional connective in the object language; by contrast, '$\vDash$' is a symbol in the *metalanguage*, i.e. it is a symbol of augmented English.

   There is an important connection between the two symbols. In classical logic, if $A \vDash B$, then $\vDash (A \supset B)$. For if $A \vDash B$, then there is no valuation of the atoms (no line on the truth-table) which makes A true and B false; but then there is no line on the truth-table for $(A \supset B)$ in which the antecedent is true and the consequent is false; so $(A \supset B)$ must be a tautology. Similarly, if $\vDash (A \supset B)$, then $A \vDash B$.

   (c) Show that $\{\vee, \neg\}$ is expressively adequate and $\{\vee, \supset\}$ isn't.

To prove the expressive adequacy of $\{\vee, \neg\}$, I will first prove the expressive adequacy of $\{\vee, \wedge, \neg\}$. That is, I need to show the following: Given any truth-table with n atoms (and so 2^n lines), I can write down a formula, D, which has that truth-table. There are two cases I need to consider:

1: *the truth-table is false on every line.* Where A is any atom which appears in the truth-table, let D be the formula $(A \wedge \neg A)$. This has the required truth-table, since it is also false on every line.

2: *at least one line of the truth-table is true.* For each true line of the truth-table, I write down the *basic conjunction* which is true on that line (and false on all others). For example: suppose I have atoms $A_1, A_2, A_3, \ldots, A_n$, and I am looking at the following line on the target truth-table:

A_1	A_2	A_3	$\ldots$	A_n	target formula
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
T	F	F	$\ldots$	T	T
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$

then I write down the conjunction '$(A_1 \wedge \neg A_2 \wedge \neg A_3 \wedge \ldots \wedge A_n)$' (being sloppy with brackets).

Having written down such a basic conjunction for all and only the *true* lines of the truth-table, I disjoin the results. That is, I now write down a long disjunction, D, where each disjunct in D is one of my basic conjunctions.

A disjunction is true on exactly those lines where at least one of the disjuncts is true, and false on all other lines. So D is true on exactly those lines where one of the basic conjunctions is true, and false everywhere else. But I have written down a basic conjunction for all and only those line where the target formula is to be true. So D has the required truth-table.

Accordingly, no matter what truth-table I am given, I can produce a formula, D, which has that truth-table. This shows that $\{\vee, \wedge, \neg\}$ is expressively adequate.

It is now easy to show that $\{\vee, \neg\}$ is expressively adequate. Observe that $\{\vee, \neg\}$ can express conjunction:

A	B	$\neg(\neg A \vee \neg B)$
T	T	T
T	F	F
F	T	F
F	F	F

So now, wherever my formula D contained a conjunction $(A \wedge B)$, for any (complex) wffs A and B, I can replace that conjunction with $\neg(\neg A \vee \neg B)$. The resulting formula will have exactly the same truth-table as D. **So $\{\vee, \neg\}$ is expressively adequate.**

Last, I need to show that $\{\vee, \supset\}$ is *not* expressively adequate. It suffices to show that there is some truth-table which cannot be expressed using only '$\vee$' and '$\supset$'. For any wffs B and C, both $(B \vee C)$ and $(B \supset C)$ are true whenever B and C are true. So, for any formula whose only logical constants are '$\vee$' and '$\supset$', the top line of the truth table for that formula must be true. But then it is impossible to express negation:

A	$\neg A$
T	F
F	T

since the first line of this truth table must be false. **So $\{\vee, \supset\}$ is not expressively adequate.**

2 Attempt all parts of this question.

(a) Using the following translation manual:
'a' denotes Abelard
'e' denotes Eloise
'f' denotes Fulbert
'Sx' expresses: x is a student
'Cx' expresses: x is in a convent
'Px' expresses: x is a philosopher
'Lxy' expresses: x loves y
Taking the domain to be all people, translate the following into QL$^=$

(i) Not every student in a convent is a philosopher.
$\neg\forall x((Sx \wedge Cx) \supset Px)$

(ii) Eloise loves some philosopher only if all students are philosophers.
$(\exists x(Px \wedge Lex) \supset \forall x(Sx \supset Px))$

(iii) Anyone who loves no philosophers does not love Abelard.
$\forall x(\neg\exists y(Py \wedge Lxy) \supset \neg Lxa)$

(iv) Eloise loves at most one philosopher.
$\forall x\forall y([(Px \wedge Lex) \wedge (Py \wedge Ley)] \supset x = y)$

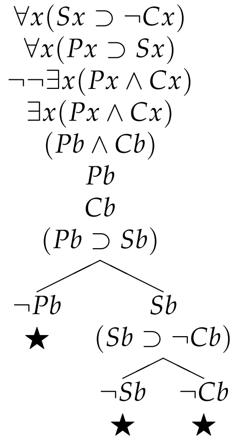
(v) There are exactly two students whom Fulbert loves.
$\exists x\exists y([(Sx \wedge Lfx) \wedge (Sy \wedge Lfy)] \wedge [\neg x = y \wedge \forall z((Sz \wedge Lfz) \supset (x = z \vee y = z))])$

(vi) If Eloise is in a convent, then Eloise is the only person Fulbert does not love; otherwise, the only person Fulbert does not love is Abelard.
$([Ce \supset \forall x(\neg Lfx \equiv x = e)] \wedge [\neg Ce \supset \forall x(\neg Lfx \equiv x = a)])$

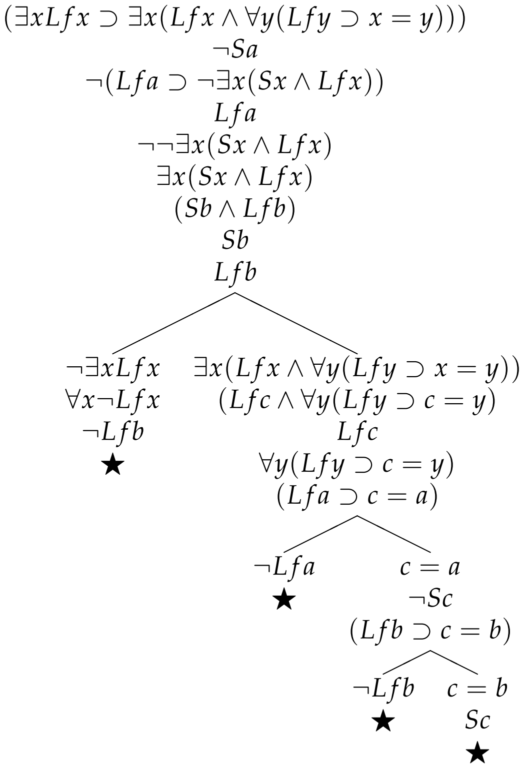
(vii) If exactly two philosophers are in a convent, then one of them is Eloise.
$\forall x\forall y(\{[(Px \wedge Cx) \wedge (Py \wedge Cy)] \wedge [\neg x = y \wedge \forall z((Pz \wedge Cz) \supset (x = z \vee y = z))]\} \supset \{x = e \vee y = e\})$

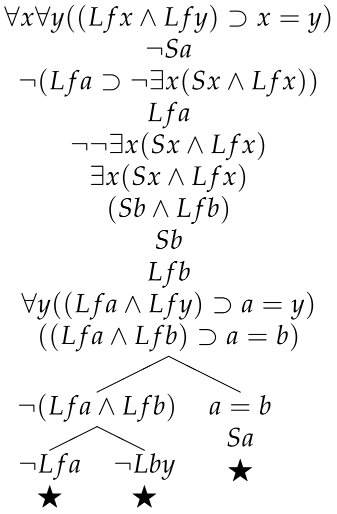
(b) Using the same translation manual, render the following arguments into QL$^=$ and use trees to show that they are valid.

(i) No student is in a convent. The only philosophers there are are also students. So no philosopher is in a convent.



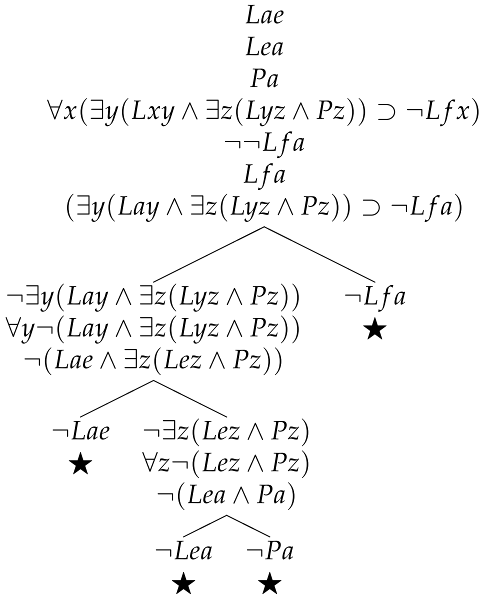
(ii) If Fulbert loves anyone, he loves exactly one person. Abelard is not a student. So if Fulbert loves Abelard, Fulbert loves no students.



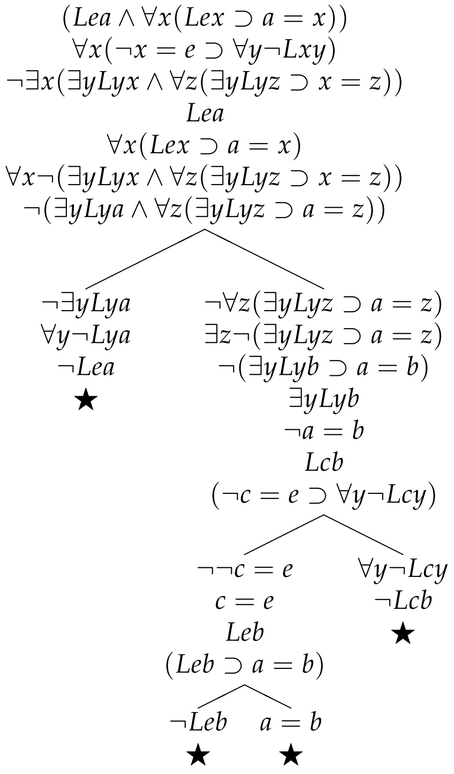


The shorter tree follows from realising that the first premiss amounts to 'Fulbert loves at most one person'.

(iii) Abelard loves Eloise. Eloise loves Abelard. Abelard is a philosopher. Fulbert loves no one who loves anyone who loves a philosopher. So Fulbert does not love Abelard.



(iv) Eloise loves Abelard and only Abelard. No one else loves anyone. So exactly one person is loved.


$(Lea \wedge \forall x(Lex \supset a = x))$
$\forall x(\neg x = e \supset \forall y \neg Lxy)$
$\neg \exists x(\exists y Lyx \wedge \forall z(\exists y Lyz \supset x = z))$
Lea
$\forall x(Lex \supset a = x)$
$\forall x \neg(\exists y Lyx \wedge \forall z(\exists y Lyz \supset x = z))$
$\neg(\exists y Lya \wedge \forall z(\exists y Lyz \supset a = z))$
$\neg \exists y Lya$
$\forall y \neg Lya$
$\neg Lea$
★
$\neg \forall z(\exists y Lyz \supset a = z)$
$\exists z \neg(\exists y Lyz \supset a = z)$
$\neg(\exists y Lyb \supset a = b)$
$\exists y Lyb$
$\neg a = b$
Lcb
$(\neg c = e \supset \forall y \neg Lcy)$
$\neg\neg c = e$
$c = e$
Leb
$(Leb \supset a = b)$
$\neg Leb$
★
$a = b$
★
$\forall y \neg Lcy$
$\neg Lcb$
★

3. (a) Define what it means to say that:

i. A binary relation R is reflexive.
R is reflexive iff $\forall x Rxx$

ii. A binary relation R is symmetric.
R is symmetric iff $\forall x \forall y (Rxy \supset Ryx)$

iii. A binary relation R is transitive.
R is transitive iff $\forall x \forall y \forall z ((Rxy \wedge Ryz) \supset Rxz)$

iv. A binary relation R is an equivalence relation.
R is an equivalence relation iff R is symmetric, reflexive and transitive.

(b) Say that a binary relation R is circular iff $\forall x \forall y \forall z ((Rxy \wedge Ryz) \supset Rzx)$. With this definition to hand, prove the following claims, for any binary relation R. You may use **QL** trees to prove these claims, or an informal argument.

i. Suppose R is circular and symmetric. Suppose also that every object bears R to something. Then R is reflexive.
Arbitrarily fix x and y. Since R is circular, $((Rxy \wedge Ryx) \supset Rxx)$. Since R is symmetric, $(Rxy \supset Ryx)$. So $(Rxy \supset Rxx)$. But then it follows, from the assumption, that Rxx. Since x was arbitrary, R is symmetric.

ii. Suppose R is symmetric. Then R is circular iff R is transitive.
Suppose R is symmetric and that $(Rxy \wedge Ryz)$.
First: if R is circular, then Rzx; by symmetry, Rxz; so we have $((Rxy \wedge Ryz) \supset Rxz)$. Generalising, we have transitivity.
Second: if R is transitive, then Rxz; by symmetry, Rzx; so we have $((Rxy \wedge Ryz) \supset Rzx)$. Generalising, we have circularity.

iii. R is reflexive and circular iff R is an equivalence relation.
First: suppose R is reflexive and circular. To prove the relation is equivalent, we need to demonstrate symmetry and transitivity. To show R is symmetric, note that, by circularity, $((Rxx \wedge Rxz) \supset Rzx)$. Since R is reflexive, we invariably have Rxx. Hence $(Rxz \supset Rzx)$; generalising, R is symmetric. Since R is circular and symmetric, it is transitive (see answer to previous question). So R is reflexive, symmetric and transitive, i.e. R is an equivalence relation.
Second: suppose R is an equivalence relation. Then R is symmetric and transitive; so R is circular (by the answer to the previous question).

(c) Let the domain of quantification be all people alive today. For each of the following relations, say whether it is (1) reflexive, (2) symmetric, (3) transitive, and (4) circular. Where the answer is 'no', or a case could be made either way, explain your answer.

i. x and y share both parents.
Reflexivity can be argued both ways. If we formalise this as 'there are distinct s and t, and s and t are both parents of x and y (and the only such parents)', then R is not reflexive: consider the case of someone who has no parents, or more than two parents. If we treat it as a primitive, though, we are likely to regard it as reflexive.
Symmetric
Transitive
Circular

ii. x and y are both female and share both parents.
Not Reflexive: there are some men.
Symmetric
Transitive
Circular

iii. x is female and shares both parents with y.
Not Reflexive: there are some men.
Not Symmetric: consider a woman a and her brother b; then Rab but $\neg Rba$, since b is not female.
Transitive.
Not Circular: again, consider a woman a and her brother b; then $(Raa \wedge Rab)$ but $\neg Rba$, since b is not female. This contradicts circularity.

iv. If x is female and shares both parents with y, then y is female and shares both parents with x.

Reflexive.
Not Symmetric. Consider a man a and his sister b. Then Rab (since a is not female), but $\neg Rba$ (since b is female but a is not).
Not Transitive. Consider a woman a and her brother c. Let b be someone unrelated to a and c. Then Rab (since they are unrelated) and Rbc (since they are unrelated); but $\neg Rac$ (since a is female and c is not.)
Not Circular. With the same three people, we have Rcb and Rba, but $\neg Rac$.

4. Attempt all parts of this question.

   (a) Let A be the set of all women, B be the set of all Russians, and C be the set of all married Russians. Give the natural language translations of the following:

      i. $C \subseteq (A \cap B)$
      The set of married Russians is a subset of the intersection of the set of Russians and the set of women. Thus: every married Russian is a Russian woman (i.e., no Russian men are married).

      ii. Alexandra $\in (B \cup A)$
      Alexandra is a member of the union of the set of Russians and the set of women. Thus: Alexandra is either Russian or a woman.

      iii. $C \subset \wp(B)$
      The set of married Russians is a proper subset of the powerset of the set of Russians. Thus: not every Russian is married.

      iv. $(A \cap B) \neq \emptyset$
      The intersection of the set of Russians and the set of women is not the empty set. Thus: there is at least one Russian woman.

      v. Tatjana $\in (A/B)$
      Tatjana is a member of the set of women who are not Russian. Thus: Tatjana is a non-Russian woman.

      vi. $\{x : x \in A\}$
      The set of women. (Note: this is a name, note a sentence.)

   (b) What is the axiom of extensionality?

   The axiom of extensionality is the set theoretic axiom which governs the identity of sets. The axiom states that two sets are identical if and only if they have all of their members in common; in symbols:

$$(\forall x)(\forall y)(x = y \supset \forall z(z \in x \equiv z \in y))$$

   where the quantifiers range over sets.

   (c) Suppose that X = {Ringo, John, Paul, George}. And suppose that all and only the members of X are groovy. Show:

      i. That there is no set of all the non-groovy things.
      Assume the existence of such a set, Q. Since X has as its members all the groovy things, Q must contain everything that it is not in X. So $X \cup Q$ is the universal set. But there is no universal set; so Q does not exist.
      (To show that there is no universal set, suppose, on the contrary, that V is the universal set. By the Axiom of Separation, $R = \{x : x \in V \wedge \neg x \in x\} = \{x : \neg x \in x\}$ exists. This is the Russell set: if $R \in R$, then $\neg R \in R$, by the definition of R; but if $\neg R \in R$, then $R \in R$, again by the definition of R. So $R \in R$ iff $\neg R \in R$, which is a contradiction. So the universal set does not exist.)

      ii. That no member of $\wp(X)$ is groovy.
      This question is poorly phrased. If we knew that John, Ringo, etc. are not sets, it would be easy to show that no member of $\wp(X)$ is groovy. By the definition of powerset, $\wp(X) = \{x : x \subseteq X\}$, the members of $\wp(X)$ are sets. Now, if there were some z such that $z \in \wp(X)$ and $z \in \wp(Y)$, then $z \in X$. But since we are assuming that no element of X is a set, there can be no such z.
      However, the question has *not* told us that the elements of X are not sets. For it might be that 'John' refers to the {Ringo}. In which case, John is a member both of X and $\wp(X)$. So we cannot prove (ii).

      iii. That if Yoko is a subset of X then Yoko either has a groovy member or is the empty set.
      If Yoko is a subset of X then every x in Yoko is in X. Since X contains *only* groovy things, Yoko contains only groovy things. So anything in Yoko is groovy. So either Yoko contains something groovy, or Yoko is the empty set.

(d) What is Bayes' Theorem?

Bayes's Theorem is a law of probability connecting conditional probabilities. Where:

- $P(A)$ is the unconditional probability of event A;
- $P(B)$ is the unconditional probability of event B, and $P(B) \neq 0$; and
- $P(A|B)$ is the conditional probability of A given B (and conversely for $P(B|A)$);

we have:

$$P(A|B) = \frac{P(B|A)P(B)}{P(A)}$$

(e) You are faced with two bags. Bag A contains 10 red balls, 9 of which have a black spot, and 2 unspotted white balls. Bag B contains 10 red balls, 1 of which has a black spot, and 50 unspotted white balls. You are passed one of the bags. You don't know which bag you have, though you know that there is a $\frac{1}{4}$ chance it is bag A, and a $\frac{3}{4}$ chance that it is bag B. What is:

i. The probability that you will pull a red ball out of the bag?

Let R be the event of getting a red ball, A be the event of getting bag A, and B be the event of getting bag B. Since A and B are mutually exclusive and exhaustive:

$$\begin{aligned} P(R) &= P(R \wedge A) + P(R \wedge B) \\ &= \frac{1}{4} \times \frac{10}{12} + \frac{3}{4} \times \frac{10}{60} \\ &= \frac{1}{3} \end{aligned}$$

ii. The probability that you will pull a spotted ball out of the bag, given that you have bag B?

This is just the frequency of spotted balls in bag B, namely $\frac{1}{60}$.

iii. The probability that the ball you will pull out will be spotted, given that it will be a red ball?

Let S be the event of getting a spotted ball, with other abbreviations as in question i.

$$P(S|R) = \frac{P(S \wedge R)}{P(R)}$$

We know from question i that $P(R) = \frac{1}{3}$; it remains to calculate $P(S \wedge R)$. Again, since A and B are mutually exclusive exhaustive events:

$$\begin{aligned} P(S \wedge R) &= P(S \wedge R \wedge A) + P(S \wedge R \wedge B) \\ &= \frac{9}{12} \times \frac{1}{4} + \frac{1}{60} \times \frac{3}{4} \\ &= \frac{1}{5} \end{aligned}$$

So $P(S|R) = \frac{3}{5}$.

NB: a common mistake is to assume that the answer can be calculated as:

[probability ball will be spotted given it will be red, given that you have bag A] +
[probability ball will be spotted given it will be red, given that you have bag B]

To see that this is a mistake reason as follows. The frequency of red balls in bag A is much higher than the frequency of red balls in bag B. So, since what is given is that the ball will be red, it is more *likely that you are dealing with bag A than with bag B. (Indeed, suppose that B contained* no *red balls; then it would be* certain *that you had been given bag A.)*

iv. The probability that you will first pull a white ball, followed by a red ball with a spot, given that you have bag A?

I shall assume that we do *not* replace the first ball after drawing it.

The probability that I will first pull out a white ball, given that I have bag A, is just the frequency of white balls in bag A, i.e. $\frac{1}{6}$.

The probability that I will pull out a red ball with a spot, given that I have bag A and drew a white ball with my first draw, is the frequency of red balls with spots remaining in bag A, i.e. $\frac{9}{11}$.

So the overall probability is $\frac{1}{6} \times \frac{9}{11} = \frac{3}{22}$.

v. The probability that the ball you will pull out will be white, given that it will be a spotted white ball?

Let S be the event of getting a spotted ball and W be the event of getting a white ball; so we are looking for:

$$P(W|(S \wedge W)) = \frac{P(W \wedge (S \wedge W))}{P(S \wedge W)}$$

But $P(S \wedge W) = 0$, since there are no spotted white balls. So this probability is undefined.