1A Logic — 2013 Model Answers

Section A

1. This is a question about TFL. Attempt all parts of this question. [**Note that most of the material in this question has been moved to the 'metatheory' section of the 1B logic paper, and is no longer examinable at part 1A. Answers are provided for instructional purposes only. The texbook for this part of the syllabus is available on Tim Button's website**]

(a) Explain what these three sentences mean, and explain the differences between them: [15]

(i) $A \models C$
(ii) $A \vdash C$
(iii) $A \to C$

Sentence (i) is not a sentence of TFL, rather it is a sentence of mathematically augmented English that we use to talk *about* sentences of TFL; i.e. is is *metalinguistic*, and the sentences flanking $\models$ are mentioned, rather than used. $A \models C$ says something about the valuation functions of TFL, namely that every valuation under which A is true is a valuation on which C is true. Sentence (ii) is likewise metalinguistic. However, it says something our natural deduction system, namely that there is some proof of C, the only premise of which is A. Contrasting to this, sentence (iii) is a TFL sentence, where '$\to$' is a truth–functional connective in the language of TFL (the material conditional), and 'A' and 'C' are TFL sentences.

However, these three notions are importantly connected; Our natural deduction system is sound, meaning that if $A \vdash C$ then $A \models C$. Conversely, our proof system is complete, meaning that if $A \models C$ then $A \vdash C$. Moreover, TFL has the 'deduction property', namely that $A \vdash C$ if, and only if, $\vdash A \to C$. Hence although sentences (i) through (iii) mean very different things, we have it that $A \models C$ iff $A \vdash C$ iff $\vdash A \to C$ iff $\models A \to C$.

(b) State and prove the Disjunctive Normal Form Theorem. [50]

A sentence of the language of TFL is in DNF (Disjunctive Normal Form) if and only if it satisfies the following conditions:

(i) The only connectives occurring in the sentence are negations, conjunctions and disjunctions.

(ii) Any occurrence of negation has minimal scope (i.e. is immediately prefixed to an atomic sentence).
(iii) No disjunction in the sentence appears inside the scope of any conjunction appearing in the sentence.

The DNF theorem states that every sentence has an a tautologically equivalent sentence in DNF.

Proof: For any sentence Φ of our language, let $\phi_1 ... \phi_n$ be the atomic sentences occurring in Φ. Now, examine the truth table of Φ. If every line of the table is false, then Φ is tautologically equivalent to $(P \wedge \neg P)$, which is in DNF. If Φ is true on at least one line of its truth table, construct a sentence of the following form for each line i of Φ's truth table on which Φ is true: $\Psi_i = (\psi_1 \wedge ... \wedge \psi_n)$, where $\psi_m = \phi_m$ if ϕ_m is true on line i, other wise $\psi_m = \neg \phi_m$. Clearly, each Ψ_i is true on exactly the valuation of atoms on line i of Φ's truth table (though note that a fully rigorous proof of this would require mathematical induction). Now let $\Psi = (\Psi_{i_1} \vee .... \vee \Psi_{i_m})$. By construction, Ψ is in DNF, and is tautologically equivalent to Φ, since if the latter is true, one of Ψ's disjuncts is true, and vice–versa. Since these cases are exhaustive, the DNF theorem follows.

(c) Explain what it means to say, of some connectives, that they are jointly expressively adequate. Show that '$\wedge$' and '$\neg$' are jointly expressively adequate. You may rely on your answer to part (B). [15]

Connectives are said to be jointly expressively adequate if, and only if, every sentence of the language of TFL is tautologically equivalent to a sentence in which the only those connectives appear. Our proof of the DNF theorem is sufficient to show that '$\neg$', '$\wedge$', and '$\vee$' are jointly expressively adequate. It follows via De Morgan's laws that '$\neg$' and '$\wedge$' are jointly expressively adequate. Use the procedure above, for an arbitrary sentence, of constructing a tautologically equivalent sentence in DNF. If our our original sentence was a contradiction, it's DNF equivalent is $(P \wedge \neg P)$, in which no disjunctions appear. Otherwise, it is of the form $(\Psi_{i_1} \vee .... \vee \Psi_{i_m})$. By De Morgan's laws, this is tautologically equivalent to $\neg(\neg\Psi_{i_1} \wedge .... \wedge \neg\Psi_{i_m})$, in which no disjunctions appear by the construction given above.

(d) Are the connectives '$\wedge$', '$\vee$', '$\rightarrow$' and '$\leftrightarrow$' jointly expressively adequate? Explain your answer. [20]

They are not. A fully rigorous proof of this fact would require mathematical induction. However, intuitively they are not expressively adequate because no sentence whose only connectives are amongst those given is tautologically equivalent to $\neg P$. This sentence is false if all atomic sentences appearing in it are true. However, if all of the atomic sentences appearing in a conjunction, disjunction, conditional, or biconditional sentence are true, then that sentence is also true, as an examination of the first line of the truth–tables of these connectives reveals. Induction is needed to show this in full generality, but it is clear enough that if you only have connectives that are true when all of their arguments are true, no combination of these connectives and atomic sentences will be able to express negation.

2. Attempt all parts of this question. You **must** use the proof system from the course textbook.

(a) Show each of the following: [40]

(i) $\vdash (P \to Q) \lor (Q \to P)$

$$
\begin{array}{rlll}
1 & \mid & \mid \underline{P} & \\
2 & \mid & \mid \quad \mid \underline{Q} & \\
3 & \mid & \mid \quad \mid P & \text{R, 1} \\
4 & \mid & \mid Q \to P & \to\text{I, 2–3} \\
5 & \mid & \mid (P \to Q) \lor (Q \to P) & \lor\text{I, 4} \\
6 & \mid & \mid \underline{\neg P} & \\
7 & \mid & \mid \quad \mid \underline{P} & \\
8 & \mid & \mid \quad \mid \bot & \bot\text{I, 6, 7} \\
9 & \mid & \mid \quad \mid Q & \bot\text{E, 8} \\
10 & \mid & \mid P \to Q & \to\text{I, 7–9} \\
11 & \mid & \mid (p \to Q) \lor (Q \to P) & \lor\text{I, 10} \\
12 & \mid & (P \to Q) \lor (Q \to P) & \text{TND, 1–5, 6–11}
\end{array}
$$

(ii) $\neg(P \leftrightarrow Q), \neg P \vdash Q$

1	$\neg(P \leftrightarrow Q)$	
2	$\underline{\neg P}$	
3	$\quad \mid \underline{\neg Q}$	
4	$\quad \mid \quad \mid \underline{Q}$	
5	$\quad \mid \quad \mid \bot$	$\bot$I, 3, 4
6	$\quad \mid \quad \mid P$	$\bot$E, 5
7	$\quad \mid \quad \mid \underline{P}$	
8	$\quad \mid \quad \mid \bot$	$\bot$I, 2, 7
9	$\quad \mid \quad \mid Q$	$\bot$E, 8
10	$\quad \mid P \leftrightarrow Q$	$\leftrightarrow$I, 4–6, 7–9
11	$\quad \mid \bot$	$\bot$I, 1, 10
12	$\neg\neg Q$	$\neg$I, 3–11
13	Q	DNE, 12

(b) Show each of the following: [60]

(i) $\exists x(Fx \vee Gx) \vdash \exists xFx \vee \exists xGx$

1	$\exists x(Fx \vee Gx)$	
2	$\quad \mid \underline{Fa \vee Ga}$	
3	$\quad \mid \quad \mid \underline{Fa}$	
4	$\quad \mid \quad \mid \exists xFx$	$\exists$I, 3
5	$\quad \mid \quad \mid \exists xFx \vee \exists xGx$	$\vee$I, 4
6	$\quad \mid \quad \mid \underline{Ga}$	
7	$\quad \mid \quad \mid \exists xGx$	$\exists$I, 6
8	$\quad \mid \quad \mid \exists xFx \vee \exists xGx$	$\vee$I, 7
9	$\quad \mid \exists xFx \vee \exists xGx$	$\vee$E, 2, 3–5, 6–8
10	$\exists xFx \vee \exists xGx$	$\exists$E, 1, 2–9

(ii) $\forall x(Fx \to \forall yRxy), \forall x(Gx \to \forall zRxz), \forall x(\forall wRxw \to (Fx \land Gx)) \vdash \forall x(Fx \leftrightarrow Gx)$

1	$\forall x(Fx \to \forall yRxy)$	
2	$\forall x(Gx \to \forall zRxz)$	
3	$(\forall wRxw \to (Fx \land Gx)$	
4	$\quad Fa$	
5	$\quad Fa \to \forall yRay$	$\forall$E, 1
6	$\quad \forall yRay$	$\to$E, 4, 5
7	$\quad \forall wRaw \to (Fa \land Ga)$	$\forall$E, 3
8	$\quad Rab$	$\forall$E, 6
9	$\quad \forall wRaw$	$\forall$I, 8
10	$\quad Fa \land Ga$	$\to$E, 7, 9
11	$\quad Ga$	$\land$E, 10
12	$\quad Ga$	
13	$\quad Ga \to \forall zRaz$	$\forall$E, 2
14	$\quad \forall zRaz$	$\to$E, 12, 13
15	$\quad \forall wRaw \to (Fa \land Ga)$	$\forall$E, 3
16	$\quad Rab$	$\forall$E, 14
17	$\quad \forall wRaw$	$\forall$I, 16
18	$\quad Fa \land Ga$	$\to$E, 15, 17
19	$\quad Fa$	$\land$E, 18
20	$Fa \leftrightarrow Ga$	$\leftrightarrow$I, 4–11, 12–19
21	$\forall x(Fx \leftrightarrow Gx)$	$\forall$I, 20

(iii) $\forall x \exists y Rxy, \exists x \forall y x = y \vdash \exists y \forall x Rxy$

1	$\forall x \exists y Rxy$	
2	$\exists x \forall y x = y$	
3	$\exists y Ray$	$\forall$E, 1
4	$\forall y b = y$	
5	Rac	
6	$b = c$	$\forall$E, 4
7	$b = d$	$\forall$E, 4
8	$c = d$	=E, 6, 7
9	Rad	=E, 5, 8
10	Rad	$\exists$E, 3, 5–9
11	Rad	$\exists$E, 2, 4–10
12	$\forall x Rxd$	$\forall$I, 11
13	$\exists y \forall x Rxy$	$\exists$I, 12

3. Attempt all parts of this question.

(a) Using the following symbolization key

domain: all physical objects
Mx: x is a mug
Rx: x is red
Tx: x is a table
Bxy: x belongs to y
a: Alice

symbolize each of the following sentences as best you can in FOL. If any sentences are ambiguous, or cannot be symbolized very well in FOL, explain why. [65]

(i) Every mug belonging to Alice is red.

$\forall x((Mx \wedge Bxa) \rightarrow Rx)$

(ii) The table is red.

$\exists x \forall y((Ty \leftrightarrow y = x) \wedge Rx)$

(iii) Alice's mug is red.

$$\exists x \forall y(((My \wedge Bya) \leftrightarrow x = y) \wedge Rx)$$

(iv) Alice's mug does not exist.

$$\neg \exists x \forall y((My \wedge Bya) \leftrightarrow x = y)$$

Comment: Formalizing is slightly awkward here. Because we do not have an 'exists' predicate, we cannot in FOL *say of* Alice's mug that it doesn't exist. Rather, because existence is expressed in FOL using the existential quantifier, we have to say something more akin to 'there is no such thing as Alice's mug'.

(v) Two mugs are on the table.

$$\exists x \forall y((Ty \leftrightarrow x = y) \wedge \exists v \exists w(((Mv \wedge Mw) \wedge \neg v = w) \wedge (Bvx \wedge Bwx)))$$

Comment: This is somewhat awkward because our symbolization key doesn't have a predicate 'x is on y'. Here, and later in this question, I've assumed that whatever is on a table belongs to it, and whatever isn't on a table doesn't belong to it.

(vi) If the mug belongs to anyone, it belongs to Alice.

$$\exists x \forall y((My \leftrightarrow x = y) \wedge (\exists v Bxv \rightarrow Bxa))$$

(vii) None of the mugs on the table is Alice's.

$$\exists x \forall y((Ty \leftrightarrow x = y) \wedge \neg \exists z((Mz \wedge Bzx) \wedge Bza))$$

Comment: The sentence is ambiguous in several ways. I have taken it to mean that there is a table, and it isn't the case that some mug on it is Alice's. It could also be read to mean that there is a table, there are some mugs on the table, and none of them are Alice's. Further, it could be read as meaning that there is a table, there is some mug which is Alice's, and it isn't the case that it is identical to any mug on the table. Finally it could also be taken to mean that there is a table, there is some mug belonging to Alice, there are some mugs on the table and that Alice's isn't identical to any of them.

(viii) Every mug is on exactly one table, and on every table there is exactly one mug.

$$(\forall x(Mx \rightarrow \exists y \forall z((Bxz \wedge Tz) \leftrightarrow y = z))) \wedge (\forall x(Tx \rightarrow \exists y \forall z((Bzx \wedge Mz) \leftrightarrow y = z))))$$

(b) Show that each of the following claims is **wrong**. [35]

(i) $Fa, \neg Ga, Fb, \neg Gb, \neg Fc, Gc \models \forall x(Fx \leftrightarrow \neg Gx)$

$D = \{0, 1, 2, 3\}$
$|F| = \{0, 1, 3\}$
$|G| = \{2, 3\}$
$|a| = 0$
$|b| = 1$
$|c| = 2$
$|d| = 3$

(ii) $\forall x(Fx \rightarrow \exists y(Gy \wedge Rxy \wedge \forall z((Gz \wedge Rxz) \rightarrow y = z)))) \models \forall x(Gx \rightarrow \exists y(Fy \wedge Ryx \wedge \forall z((Fz \wedge Rzx) \rightarrow y = z)))$

$D = \mathbb{N}$
$|F| = \{x|x \text{ is odd}\}$
$|G| = \{x|x \text{ is even}\}$
$|R| = \{(x, y)|(x + 1) = y\}$

In this interpretation, 0 serves as our counterexample. It is indeed true that every odd number has a unique even successor. However it is not true that of all even numbers that they have a unique odd predecessor, since no natural number is less than 0.

(iii) $\exists x\forall y\neg Rxy, \forall x\neg Rxx, \forall x\exists y(Rxy) \models \exists x\exists y(\neg x = y \wedge \exists z(Rxz \wedge Ryz))$

$D = \mathbb{N}$
$|R| = \{(x, y)|(x + 1) = y\}$

In this interpretation, all of the sentences to the left of the turnstile are true (0 is not the successor of anything, no number is its own successor, and every number has a successor). But the sentence to the right of the turnstile is false, since no distinct numbers share a successor (i.e. if $\neg x = y$ then $\neg(x + 1) = (y + 1)$).

4. Attempt all parts of this question.

(a) Write down the axiom of extensionality. Then, using standard notation, define the set-theoretic notions of: union, intersection, subset, proper subset, and power set. [10]

Throughout this question, we take upper-case variables to have only sets as their range. Lower-case variables range over everything, as usual.

Extensionality: $\forall X\forall Y(X = Y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$

$A \cup B = \{x | x \in A \vee x \in B\}$
$A \cap B = \{x | x \in A \wedge x \in B\}$
$A \subseteq B \leftrightarrow \forall x(x \in A \rightarrow x \in B)$
$A \subset B \leftrightarrow (A \subseteq B \wedge \neg A = B)$
$\mathcal{P}(A) = \{X | X \subseteq A\}$

(b) Give examples for each of the following: [10]

(i) Three non-empty sets A, B, and C, such that none of $A \cap B$, $B \cap C$, and $A \cap C$ is empty, but $(A \cap B) \cap C$ is empty. Let $A = \{0, 1\}$, $B = \{1, 2\}$ and $C = \{2, 0\}$.

(ii) Two different non-empty sets, A and B, such that $(\mathcal{P}(A) \cup \mathcal{P}(B)) = \mathcal{P}(A \cup B)$. Let $A = \{0\}$ and $B = \{0, 1\}$. Then $\mathcal{P}(A) = \{\emptyset, \{0\}\}$, $\mathcal{P}(B) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$, and $(\mathcal{P}(A) \cup \mathcal{P}(B)) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$. Further, $A \cup B = \{0, 1\} = B$, and $\mathcal{P}(B) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\} = (\mathcal{P}(A) \cup \mathcal{P}(B))$, as required.

(c) Give examples of each of the following: [25]

(i) A set whose intersection with its power set is non-empty. $\{\emptyset\}$. The intersection of this set with its powe rset is itself, which is non-empty.

(ii) A set whose intersection with the power set of its power set is non-empty. $\{\emptyset\}$. Again the intersection of this set with its power set's power set is $\{\emptyset\}$, which is non-empty.

(iii) A non-empty set that is a subset of the power set of one of its members. $\{\emptyset\}$ once again does the job. This set is identical to the power set of its only member, and hence is a subset of a member's power set.

(d) Write down the axioms of probability. Explain conditional probability. [10]

Probability Axioms:

$Pr(V) = 1$, where V is our sample space (set of possible outcomes).
$Pr(X) \geq 0$ where $X \in \mathcal{P}(V)$
if $X \cap Y = \emptyset$ then $Pr(X \cup Y) = Pr(X) + Pr(Y)$

$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$. This definition is rather intuitive; we can take $Pr(A)$ as keeping track of the A's in the sample space, and think of $Pr(A|B)$ as tracking the A's amongst the B's (which are themselves in the sample space). This is exactly what we should expect of a definition of 'the probability of A, given B'.

(e) There are two equally probable hypotheses: Either Bryce baked exactly 10 cupcakes today, or Bryce baked exactly 100 cupcakes today. In either case, Bryce piped unique numbers onto them: between 1 and 10, if there are exactly 10 cupcakes, or between 1 and 100, if there are exactly 100 cupcakes. Bryce hands you a cupcake with the number 9 piped onto it. How probable is it, now, that Bryce baked exactly 100 cupcakes today? Explain your reasoning, highlighting any assumptions that you have made. [15]

To answer this question, we will assume that '9' is the *only* numeral piped onto the cake we're given (i.e. we rule out that the cake has '39' piped onto it, for instance). We further assume that the hypotheses are exhaustive, and that Bryce handed us a cake at random. We will use:

Bayes's Theorem: $Pr(H|E) = \frac{(Pr(E|H)Pr(H))}{((PrE|H)Pr(H))+(Pr(E|\neg H)Pr(\neg H))}$

Letting 'E' mean that you have been given a cake with the number '9' on it, and H be the hypothesis that Bryce made 100 cupcakes. Since the two hypotheses are equally probable, we have:

$$Pr(H|E) = \frac{(\frac{1}{100}\times\frac{1}{2})}{(\frac{1}{100}\times\frac{1}{2})+(\frac{1}{10}\times\frac{1}{2})} = \frac{\frac{1}{200}}{(\frac{1}{200}+\frac{1}{20})} = \frac{\frac{1}{200}}{\frac{11}{200}} \approx 0.09$$

(f) Attempt both parts of this question. [30]

(i) You are tossing a fair six-sided die. What is the probability that it lands 6 on each of the first three tosses? What is the probability that it lands 1, then 2, then 5?

Our sample space for a toss of a fair die $V = \{1, 2, 3, 4, 5, 6\}$. The result of three tosses is then $V \times V \times V = V^3$. Exactly one member of V^3 corresponds to rolling a 6, then a 6, then a 6, and likewise to rolling a 1, then a 2, then a 5. These are therefore equally probable. Since there are $6 \times 6 \times 6 = 216$ members of V^3, the probability in each case is $\frac{1}{216}$.

(ii) Mr Corleone always chooses the same national lottery numbers. They came up in three successive lotteries, and now Mr Corleone is rich. But the lottery organisers are suing him for fixing the lottery. Mr Corleone's defence lawyer says: 'It is no more or less likely, that there numbers should come up three times in a row, than that any other sequence of numbers should come up; so why should it be special grounds for suspicion?' Is there anything wrong with the lawyers argument? Carefully explain your answer.

The lawyer's argument is fallacious. It is equally likely that Corleone's numbers come up three times in a row as any other particular triple of sequences, and hence, given that Corleone picks the same numbers each week, no outcome implicates Mr Corleone in fraud. But crucially, the defence has failed to take account of the probability that any given person wins the lottery in a given week. We'll use Bayes's theorem again, and let 'E' be 'Mr Corleone won the lottery three times in a row', and 'H' be 'Mr Corleone tried to fix the lottery'. The key problem with the lawyers argument is that, since winning the lottery three times in a row is so unlikely, $Pr(H|E)$ is close to 1 however trustworthy we take the suspect to be, given that $Pr(E|\neg H)$ is so close to 0. Let's spell this out in more detail.

According to *Google*, the odds of winning the national lottery are approximately 1/175,000,000. Hence, the odds of winning three times in a row are approximately 1 in 5.3×10^{25}. We'll be as generous as we can to Mr Corleone; let's assume that lottery fixing only works 1 time in 10, and that Mr Corleone is so trustworthy that the odds of him committing this heinous crime are, absent of evidence, only 1 in a trillion. Plugging this into Bayes's theorem:

$$Pr(H|E) = \frac{\frac{1}{10}\times\frac{1}{10^{12}}}{(\frac{1}{10}\times\frac{1}{10^{12}})+(\frac{5}{10^{25}}\times(\frac{10^{12}-1}{10^{12}}))} \approx \frac{\frac{1}{10^{13}}}{\frac{1}{10^{13}}+\frac{5}{10^{25}}} \approx \frac{0.0000000000001}{0.0000000000001000000000005}$$

So $Pr(H|E) \approx 0.999999999995$

So Mr Corleone is most probably guilty, and we've already been generous in our assumptions about his virtuous character and inability to fix lotteries!

5. Attempt all parts of this question.

(a) Explain what it means to say that a relation is: [5]

(i) Reflexive.

A relation, R, is reflexive with respect to a given domain if, and only if, $\forall x Rxx$.

(ii) Symmetric.

A relation, R, is symmetric with respect to a given domain if, and only if, $\forall x \forall y(Rxy \rightarrow Ryx)$.

(iii) Transitive.

A relation, R, is transitive with respect to a given domain if, and only if, $\forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz)$.

(b) Say that a relation is Euclidean iff $\forall x \forall y \forall z((Rxy \wedge Rxz) \rightarrow Ryz)$. For each of the following relations on the domain of all people (living or dead), say whether the relation is reflexive, whether it is symmetric, whether it is transitive, and whether it is Euclidean. In each case that the relation fails to have one of these properties, briefly explain your answer: [50]

(i) x and y have the same surname

This is reflexive, symmetric, transitive, and Euclidean.

(ii) x and y have the same surname or the same first name.

This is reflexive and symmetric. It is not transitive (let $x =$ Onika Maraj, $y =$ Carol Maraj, and $z =$ Carol Swain). It is also not Euclidean (let $x =$ Carol Maraj, $y =$ Onika Maraj), and $z =$ Carol Swain).

(iii) x loves y only if y loves x.

This is reflexive. It isn't symmetric (consider the case where Nicki doesn't love Meek, and Meek loves Nicki). Nor is it transitive (Suppose Nicki doesn't love Kim or Meek, Kim doesn't love Nicki or Meek, Meek doesn't love Kim, but does love Nicki, and let $x =$ Nicki, $y =$ Kim, and $z =$ Meek). Nor is it Euclidean (Let $x =$ Kim $y =$ Meek, $z =$ Nicki).

(iv) x loves y iff y loves x.

This is reflexive and symmetric. It isn't transitive (suppose as above and let $x =$ Meek, $y =$ Kim and $z =$ Nicki), nor Euclidean (let $x =$ Kim, $y =$ Meek, $z =$ Nicki)

(v) x is Winston Churchill or y is Bertrand Russell.

This isn't reflexive (it isn't true of Nicki, for example). It isn't symmetric (let $x =$ Churchill, $y =$ Nicki). It is transitive (if $x =$ Churchill, then Rxz holds for any z. If $y =$ Russell, then Ryz holds only iff $z =$ Russell, in which case Rxz for any z). It isn't Euclidean (Suppose $x =$ Churchill, $y =$ Nicki, and $z =$ Kim).

(vi) x is Winston Churchill iff y is Bertrand Russell.

This isn't reflexive (let $x =$ Churchill). It isn't symmetric ($x =$ Nicki, $y =$ Churchill). It isn't transitive ($x =$ Churchill, $y =$ Russell, and $z =$ Nicki). It isn't Euclidean either ($x =$ Russell, $y =$ Churchill, $z =$ Nicki).

Give examples of relations with the following properties. In each case be careful to specify the domain. [65]

In each case, let the domain $= \mathbb{N}$, the natural numbers.

(i) Reflexive and transitive, but not symmetric.

Consider the $\leq$ relation ('x is less–than–or–equal–to y'). This is reflexive and transitive, but not symmetric (e.g. $0 \leq 1$, but $\neg 1 \leq 0$).

(ii) Euclidean and transitive, but not reflexive.

The empty relation satisfies this trivially, since the Euclidean and transitivity criteria always have a false antecedent where R is empty. However, since there are numbers which do not bear R to themselves, the empty relation isn't reflexive.

(iii) Symmetric and transitive, but not reflexive.

The empty relation satisfies this for similar reasons.

(iv) Reflexive and symmetric, but neither transitive nor Euclidean.

Let $R = \{(x, y)|x = y \vee x = y + 1 \vee x + 1 = y\}$. This is reflexive since every number is self-identical, and symmetric, since if $x = y + 1$ then $y + 1 = x$ and vice–versa. But it isn't transitive, since $R01$ and $R12$ and $\neg R02$, nor is it Euclidean since $R10$ and $R12$ and $\neg R02$.

(v) Neither reflexive, symmetric, transitive, nor Euclidean.

Let $R = \{(0, 1), (1, 2), (0, 3)\}$. This relation may be a little gerrymandered, but it isn't reflexive (since $\neg R00$), nor is it symmetric (since $R01$ and $\neg R10$), nor transitive (since $R01$, $R12$ and $\neg R02$), nor Euclidean (since $R01$, $R03$ and $\neg R13$).