SETS, RELATIONS AND PROBABILITY LECTURE 7

1. We now consider probabilities of sets of outcomes of *repeated* tests of devices or arrangements with random results. Suppose that we have a reference set V whose members are the possible outcomes of some trial, and we wish to repeat the experiment once. Then if the possible outcomes as the same in both cases, we may consider the repeated test as *one* trial with each *ordered pair* of V corresponding to an element of the new reference set, written $V \times V$ or $V^2 = \{(x, y): x \in V \wedge y \in V\}$ (recall lecture 2 no. 3).

2. First we consider the case where the outcome of each individual trial is *probabilistically independent* of the outcome of any other. Thus suppose that you toss a coin twice, and suppose that the tosses are probabilistically independent, so that your confidence in the result of any toss is completely unaffected by your knowledge of the outcome of any distinct toss. (This is in fact something of a fiction.) Then the outcomes of each toss are elements of V = {H, T}. And the outcome of both is an element of V^2 = {(H, H), (H, T), (T, H), (T, T)}. Hence there are 4 possible outcomes: ignoring the parentheses we can write these as: HH, HT, TH, TT. Each outcomes gets probability ¼. What is the probability that you get heads at least once? It is P ({HH, HT, TH}) = ¼ + ¼ + ¼ = ¾.

3. Now consider a case of repeated trials where the outcome of each is *probabilistically dependent* on that of the others e.g. random draws from a pack of cards without replacement. Consider the case where two draws are made. Then our initial reference set V has 52 members representing one draw. The new reference set is not V^2 because you can't remove the same card twice. Instead it is $V^2 - X$, where $X = \{(x, x): x \in V\}$. That set has 51×52 members each of which (i.e. each of whose one-element subsets) has equal probability. Then what is the probability that one of them is an ace? Consider the following sets of events:

   A: The first is an ace and the second is not: 4×48 elements
   B: The second is an ace and the first is not: 4×48 elements
   C: They are both aces: 4×3 elements

   Then what we want is Pr $(A \cup B \cup C)$. We know that Pr $(A \cup B \cup C)$ = Pr (A) + Pr (B) + Pr (C) by axiom (iii), since any two of A, B, C have an empty intersection. So the answer is:

   $$\Pr (A \cup B \cup C) = (4 \times 48 + 4 \times 48 + 4 \times 3) / (51 \times 52)$$
   $$= 99 / (13 \times 51) = 33 / (13 \times 17) = 33 / 221$$

4. We can also calculate *conditional* probabilities of results of repeated trials. Suppose that we are tossing a fair coin three times. Then there will be 8 outcomes of equal probability viz HHH, HHT, HTH, HTT, THH, THT, TTH, TTT. What is the probability that they are all heads given that one of them is heads? We want:

   Pr ({HHH} | {HHH, HHT, HTH, HTT, THH, THT, TTH})

$= \Pr(\{HHH\} \cap \{HHH, HHT, HTH, HTT, THH, THT, TTH\}) / (7/8)$
$= (1/8) / (7/8) = 1/7$

5. Now suppose we want to calculate the probability that they are all heads given that the *first* toss lands heads. In this case we want:

   $\Pr(\{HHH\} \mid \{HHH, HHT, HTH, HTT\}) = (1/8) / (4/8) = 1/4$

   Notice that this answer is about twice as large as the last one. But how can this be? The results of the tosses are completely (i.e. both causally and probabilistically) independent, so if somebody notices that one of the three tosses landed heads, why should the additional information that the toss she noticed happened to be the *first* one—why should that make any difference to what you think happened on the other two? And yet statistics vindicate these probabilities: if you took a very large population of three-toss trials, about 1/7 of those in which one toss landed heads would be HHH, and about ¼ of those in which the *first* toss landed heads would be HHH. This illustrates the sensitivity of conditioning.

6. Now try calculating the probability that if two cards are drawn from a pack without replacement, both are aces given that one of them is an ace. Let C say that they are both aces and let X say that one of them is an ace. Then we already know that:

   $\Pr(C) = 12 / (51 \times 52) = 1 / 221$
   $\Pr(X) = 33 / 221$

   We also know that $C \cap X = C$ and so:

   $\Pr(C \mid X) = \Pr(C \cap X) / \Pr(X) = \Pr(C) / \Pr(X) = 1/33$

7. Now try working out the probability that they are both aces given that one of them is *the ace of spades*. Let Y say that one of them is the ace of spades and consider the following sets:

   Y_1: The first is the ace of spades (and the second is not): 51 outcomes
   Y_2: The second is the ace of spades (and the first is not): 51 outcomes
   $C \cap Y_1$: First is ace of spades, second is another ace: 3 outcomes
   $C \cap Y_2$: Second is ace of spades, first is another ace: 3 outcomes

   Hence:

   $\Pr(Y) = \Pr(Y_1) + \Pr(Y_2) = 102 / (51 \times 52)$
   $\Pr(C \cap Y) = \Pr(C \cap Y_1) + \Pr(C \cap Y_2) = 6 / (51 \times 52)$; hence
   $\Pr(C \mid Y) = 6 / 102 = 1 / 17$.

   Here too the conditional probability is highly sensitive to what is being conditionalized on. Can you explain why the answer to 7 is nearly twice the answer to 6?