## 1A Logic - 2016 Model Answers

## Section A

1. (a) Show each of the following using the proof system for TFL described in forallx.
(i) $\neg(P \leftrightarrow Q) \vdash \neg(Q \leftrightarrow P)$

| 1 | $\neg(P \leftrightarrow Q)$ |  |
| :---: | :---: | :---: |
| 2 | $Q \leftrightarrow P$ |  |
| 3 | $P$ |  |
| 4 | $Q$ | $\perp \mathrm{E}, 2,3$ |
| 5 | $Q$ |  |
| 6 | $P$ | $\perp \mathrm{E}, 2,5$ |
| 7 | $P \leftrightarrow Q$ | $\leftrightarrow \mathrm{I}, 3-4,5-6$ |
| 8 | $\perp$ | $\perp \mathrm{I}, 7,1$ |
| 9 | $\neg(Q \leftrightarrow P)$ | $\neg \mathrm{I}, 2-7$ |

(ii) $P \wedge \neg Q) \vee R \vdash Q \rightarrow(P \vee \neg \neg R)$

| 1 | $P \wedge \neg Q) \vee R$ |  |
| :---: | :---: | :---: |
| 2 | $Q$ |  |
| 3 | $P \wedge \neg Q$ |  |
| 4 | $P$ | $\wedge \mathrm{E}, 3$ |
| 5 | $P \vee \neg \neg R$ | VI, 4 |
| 6 | $R$ |  |
| 7 | $\neg R$ |  |
| 8 | $\perp$ | คI, 6, 7 |
| 9 | $\neg \neg R$ | $\neg \mathrm{I}, 7-8$ |
| 10 | $P \vee \neg \neg R$ | VI, 9 |
| 11 | $P \vee \neg \neg R$ | VE, 1, 3-5, 6-10 |
| 12 | $Q \rightarrow(P \vee \neg \neg R)$ | $\rightarrow \mathrm{I}, 2-11$ |

(iii) $\neg((P \rightarrow \neg Q) \vee \neg R) \vdash Q$

| 1 | $\neg((P \rightarrow \neg Q) \vee \neg R)$ |  |
| :---: | :---: | :---: |
| 2 | $\neg(P \rightarrow \neg Q) \wedge \neg \neg R$ | DeM, 1 |
| 3 | $\neg(P \rightarrow \neg Q)$ | $\wedge E, 2$ |
| 4 | $\neg Q$ |  |
| 5 | $P$ |  |
| 6 | $\neg Q$ | R, 4 |
| 7 | $P \rightarrow \neg Q$ | $\rightarrow \mathrm{I}, 5-6$ |
| 8 | $\perp$ | $\perp \mathrm{I}, 7,3$ |
| 9 | $\neg \neg Q$ | $\neg \mathrm{I}, 4-8$ |
| 10 | $Q$ | DNE, 9 |

(iv) $\neg(P \rightarrow(Q \vee R)) \vdash P \wedge \neg R$

| 1 | $\neg(P \rightarrow(Q \vee R))$ |  |
| :---: | :---: | :---: |
| 2 | $R$ |  |
| 3 | $P$ |  |
| 4 | $Q \vee R$ | VI, 2 |
| 5 | $P \rightarrow(Q \vee R)$ | $\rightarrow \mathrm{I}, 3-4$ |
| 6 | $\perp$ | $\perp \mathrm{I}, 5,1$ |
| 7 | $\neg R$ | $\neg \mathrm{I}, 2-6$ |
| 8 | $\neg P$ |  |
| 9 | $P$ |  |
| 10 | $\perp$ | $\perp \mathrm{I}, 9,8$ |
| 11 | $Q \vee R$ | VI, 10 |
| 12 | $P \rightarrow(Q \vee R)$ | $\rightarrow \mathrm{I}, 9-11$ |
| 13 | $\perp$ | $\perp \mathrm{I}, 12,1$ |
| 14 | $\neg \neg P$ | $\neg \mathrm{I}, 8-13$ |
| 15 | $P$ | DNE, 14 |
| 16 | $P \wedge \neg R$ | $\wedge \mathrm{I}, 15,7$ |

$(\mathrm{v}) \vdash P \leftrightarrow((P \wedge \neg Q) \vee(P \wedge Q))$

| 1 | P |  |
| :---: | :---: | :---: |
| 2 | $Q$ |  |
| 3 | $P \wedge Q$ | $\wedge \mathrm{I}, 1,2$ |
| 4 | $(P \wedge \neg Q) \vee(P \wedge Q)$ | VI, 3 |
| 5 | $\neg Q$ |  |
| 6 | $P \wedge \neg Q$ | $\wedge \mathrm{I}, 1,5$ |
| 7 | $(P \wedge \neg Q) \vee(P \wedge Q)$ | VI, 6 |
| 8 | $(P \wedge \neg Q) \vee(P \wedge Q)$ | TND, 2-4, 5-7 |
| 9 | $(P \wedge \neg Q) \vee(P \wedge Q)$ |  |
| 10 | $P \wedge \neg Q$ |  |
| 11 | $P$ | $\wedge \mathrm{E}, 10$ |
| 12 | $P \wedge Q$ |  |
| 13 | $P$ | $\wedge \mathrm{E}, 12$ |
| 14 | $P$ | VE, 9, 10-11, 12-13 |
| 15 | $P \leftrightarrow((P \wedge \neg Q) \vee(P \wedge Q))$ | $\leftrightarrow \mathrm{I}, 1-8,9-14$ |

(b) Show each of the following using the proof system for FOL described in forallx. [50]
(i) $\forall x \forall y x=y \vdash \neg \exists x \exists y \neg x=y$

| 1 | $\forall x \forall y x=y$ |  |
| :---: | :---: | :---: |
| 2 | $\exists x \exists y \neg x=y$ |  |
| 3 | $\exists y \neg a=y$ |  |
| 4 | $\neg a=b$ |  |
| 5 | $\forall y a=y$ | $\forall \mathrm{E}, 1$ |
| 6 | $a=b$ | $\forall \mathrm{E}, 5$ |
| 7 | $\perp$ | $\perp \mathrm{I}, 6,4$ |
| 8 | $\perp$ | ヨE, 3, 4-7 |
| 9 | $\perp$ | ЭE, 2, 3-8 |
| 10 | $\neg \exists x \exists y \neg x=y$ | $\neg \mathrm{I}, 2-9$ |

(ii) $\forall x \neg R x x, R a b \vdash \exists x \exists y \neg x=y$

| 1 | $\forall x \neg R x x$ |  |
| :---: | :---: | :---: |
| 2 | Rab |  |
| 3 | $a=b$ |  |
| 4 | Raa | =E, 3,2 |
| 5 | $\neg$ Raa | $\forall \mathrm{E}, 1$ |
| 6 | $\perp$ | $\perp \mathrm{I}, 4,5$ |
| 7 | $\neg a=b$ | $\neg \mathrm{I}, 3-6$ |
| 8 | $\exists y \neg a=y$ | $\exists \mathrm{I}, 7$ |
| 9 | $\exists x \exists y \neg x=y$ | $\exists \mathrm{I}, 8$ |

(iii) $\forall x \exists y(R y x \vee Q y x) \vdash \forall y(\exists x R x y \vee \exists x Q x y)$

| 1 | $\forall x \exists y(R y x \vee Q y x)$ |  |
| :---: | :---: | :---: |
| 2 | $\exists y(R y a \vee Q y a)$ | $\forall \mathrm{E}, 1$ |
| 3 | $R b a \vee Q b a$ |  |
| 4 | Rba |  |
| 5 | $\exists x R x a$ | $\exists \mathrm{I}, 4$ |
| 6 | $\exists x R x a \vee \exists x Q x a$ | VI, 5 |
| 7 | $Q b a$ |  |
| 8 | $\exists x Q x a$ | VI, 7 |
| 9 | $\exists x R x a \vee \exists x Q x a$ | VI, 8 |
| 10 | $\exists x R x a \vee \exists x Q x a$ | VE, 3, 4-6, 7-9 |
| 11 | $\exists x R x a \vee \exists x Q x a$ | $\exists \mathrm{E}, 2,3-10$ |
| 12 | $\forall y(\exists x R x y \vee \exists x Q x y)$ | $\forall \mathrm{I}, 11$ |

(iv) $\forall x \exists y R x y, \exists x \exists y R x y \vdash \exists x \exists y \exists z(R x y \wedge R y z)$

| 2 | $\forall x \exists y R x y$ $\exists x \exists y R x y$ |  |
| :---: | :---: | :---: |
| 3 | $\exists y$ Ray |  |
| 4 | Rab |  |
| 5 | $\exists y R b y$ | $\forall \mathrm{E}, 1$ |
| 6 | Rbc |  |
| 7 | $R a b \wedge R b c$ | $\wedge \mathrm{I}, 4,6$ |
| 8 | $\exists z(R a b \wedge R b z)$ | $\exists \mathrm{I}, 7$ |
| 9 | $\exists z(R a b \wedge R b z)$ | $\exists \mathrm{E}, 5,6-8$ |
| 10 | $\exists y \exists z(R a y \wedge R y z)$ | $\exists \mathrm{I}, 9$ |
| 11 | $\exists y \exists z($ Ray $\wedge$ Ryz $)$ | $\exists \mathrm{E}, 3,4-10$ |
| 12 | $\exists x \exists y \exists z(R x y \wedge R y z)$ | $\exists \mathrm{I}, 11$ |
| 13 | $\exists x \exists y \exists z(R x y \wedge R y z)$ | $\exists \mathrm{E}, 2,3-12$ |

(v) $\exists x(F x \wedge \forall y(F y \rightarrow y=x)), \forall x(F x \leftrightarrow G x) \vdash \exists x(G x \wedge \forall y(G y \rightarrow y=x))$

| 1 2 | $\begin{aligned} & \exists x(F x \wedge \forall y(F y \rightarrow y=x)) \\ & \forall x(F x \leftrightarrow G x) \end{aligned}$ |  |
| :---: | :---: | :---: |
| 3 | $F a \wedge \forall y(F y \rightarrow y=a)$ |  |
| 4 | $\forall y(F y \rightarrow y=a)$ | $\wedge \mathrm{E}, 3$ |
| 5 | Fa | $\wedge \mathrm{E}, 3$ |
| 6 | $F a \leftrightarrow G a$ | $\forall \mathrm{E}, 2$ |
| 7 | Ga | $\perp \mathrm{E}, 6,5$ |
| 8 | $G b$ |  |
| 9 | $F b \leftrightarrow G b$ | $\forall \mathrm{E}, 2$ |
| 10 | $F b$ | $\perp \mathrm{E}, 9,8$ |
| 11 | $F b \rightarrow b=a$ | $\forall \mathrm{E}, 4$ |
| 12 | $b=a$ | $\rightarrow$ E, 11, 10 |
| 13 | $G b \rightarrow b=a$ | $\rightarrow \mathrm{I}, 8-12$ |
| 14 | $\forall y(G y \rightarrow y=a)$ | $\forall \mathrm{I}, 13$ |
| 15 | $G a \wedge \forall y(G y \rightarrow y=a)$ | $\wedge \mathrm{I}, 7,14$ |
| 16 | $\exists x(G x \wedge \forall y(G y \rightarrow y=x))$ | $\exists \mathrm{I}, 15$ |
| 17 | $\exists x(G x \wedge \forall y(G y \rightarrow y=x))$ | $\exists \mathrm{E}, 1,3-16$ |

2. Attempt all parts of this question.
(a) Using the following symbolization key
domain: the Muppets
$F x: x$ is a frog
$J x: x$ tells jokes
$S x: x$ is a singer
Fxy: $x$ is funnier than $y$
Lxy: $x$ loves $y$
$g$ : Gonzo
$k$ : Kermit
$m$ : Miss Piggy
symbolize each of the following sentences as best you can in FOL, commenting on any difficulties or limitations you encounter [50]
(i) Miss Piggy loves Kermit unless Gonzo is funnier than Kermit.

## $L m k \vee F g k$

Comment: this assumes that the correct rendering of 'unless' is $\vee$.
(ii) The frog whom Miss Piggy loves also tells jokes.

$$
\exists x(F x \wedge K m x \wedge \forall y((F y \wedge K m y) \rightarrow y=x) \wedge J x)
$$

Comment: this assumes Russell's theory of descriptions.
(iii) Precisely two singers are loved by Kermit.

$$
\exists x \exists y(S x \wedge S y \wedge L k x \wedge L k y \wedge \neg x=y \wedge \forall z((S z \wedge L k z) \rightarrow(z=x \vee z=y)))
$$

Comment: Given the domain, this 'says' that precisely two singers who are also Muppets are loved by Kermit.
(iv) There are at most three singers, who tell jokes, and none of them are frogs.

$$
\begin{aligned}
& \forall w \forall x \forall y \forall z((S w \wedge S x \wedge S y \wedge S z) \rightarrow((w=x \vee w=y \vee w=z \vee x=y \vee x= \\
& z \vee y=z) \wedge J w \wedge J x \wedge J y \wedge J z \wedge \neg F w \wedge \neg F x \wedge \neg F y \wedge \neg F z))
\end{aligned}
$$

Comment: Again, we've expressed that there are at most three singers who are Muppets.
(v) A frog's jokes are always funnier than a singer's.

$$
\forall x \forall y((F x \wedge S y \wedge J x \wedge J y) \rightarrow F x y)
$$

Comment: We've captured 'every joke-telling frog is funnier than every joke-telling singer'. There are at least three limitations here. First, the original is ambiguous between 'there is some frog and some singer such that the jokes of the first are funnier than those of the second' and 'every frog has funnier jokes than every singer'. Second, we don't have the resources here to compare the funniness of jokes, only Muppets. Third, we cannot represent the 'always'.
(vi) Both singers and frogs tell jokes

$$
\exists x \exists y(S x \wedge F y \wedge J x \wedge J y)
$$

Comment: The original is multiply ambiguous. It could be read in at least the following ways: 'there is some singer who tells jokes and some frog who tells jokes' or 'all singers and all frogs tell jokes' or 'there is some particular pair of singers who tell jokes and some particular pair of frogs who tell jokes'.
(vii) All frogs love Miss Piggy, and some love Gonzo too.

$$
\forall x(F x \rightarrow L x m) \wedge \exists x(F x \wedge L x g)
$$

Comment: The second conjunct of the original is ambiguous between 'some Muppets love Gonzo too' and 'some frogs love Gonzo too'.
(viii) Given that Kermit tells jokes, some frog must tell jokes.

$$
\exists x(F x \wedge J x) \rightarrow J k
$$

Comment: There are two problems here. First, we don't have a canonical symbolization for 'given that'. Let's say that ' $A$ given that $B$ ' gets symbolized as ' $B \rightarrow A$ '. Second, the original contains the modal expression 'must', which we cannot represent in FOL.
(ix) Some frog and some singer are exactly as funny as Gonzo.

$$
\exists x \exists y(F x \wedge S y \wedge \neg F x g \wedge \neg F y g \wedge \neg F g x \wedge \neg F g y)
$$

Comment: We can't directly represent ' $x$ is exactly as funny as $y$ ' so the best we can do is represent ' $x$ isn't funnier than $y$ and $y$ isn't funnier than $x$ '.
(x) Three singing frogs are funnier than Miss Piggy.
$\exists x \exists y \exists z(F x \wedge S x \wedge F y \wedge S y \wedge F z \wedge S z \wedge \neg x=y \wedge \neg x=z \wedge \neg y=z \wedge F x m \wedge$ $F y m \wedge F z m)$

Comment: The original is ambiguous between 'there are at least three' and 'there are exactly three'. We've gone with the former.
(b) Show that each of the following claims is true. [40]
(i) $\forall y(\exists x R y x \wedge \exists x Q y x) \mid \vDash \forall y \exists x(R y x \wedge Q y x)$
$D=\{1,2,3\}$
$|R|=\{\langle 1,2\rangle,\langle 2,1\rangle\}$
$|Q|=\{\langle 1,3\rangle,\langle 3,1\rangle,\langle 3,2\rangle,\langle 2,3\rangle\}$
(ii) $\exists x(P x \rightarrow Q x) \not \models \exists x P x \rightarrow \exists x Q x$
$D=\{1,2\}$
$|P|=\{2\}$
$|Q|=\{ \}$
(iii) $\forall x(P x \rightarrow \exists y(R x y \wedge R y x)), \neg \exists x R x x \not \vDash \forall x \neg P x$
$D=\{1,2\}$
$|P|=\{1\}$
$|R|=\{\langle 1,2\rangle,\langle 2,1\rangle\}$
(iv) $\exists x \forall y(F y \leftrightarrow x=y), \exists x \forall y(G y \leftrightarrow x=y) \not \vDash \exists x \exists y(\neg x=y \wedge \forall z((F z \vee G z) \leftrightarrow$ $(x=z \vee y=z)))$
$D=\{1\}$
$|F|=\{1\}$
$|G|=\{1\}$
(c) Explain the differences between what is symbolized by ' $\vdash$ ' and ' $\vDash$ '. [10]
' $\vdash$ ' expresses syntactic or proof-theoretic consequence: ' $\Gamma \vdash \phi$ ' means that there is a proof of $\phi$ with no open assumptions but $\Gamma$ using only the rules of a given deductive system. ' $\models$ ' expresses semantic or model-theoretic consequence: ' $\Gamma \models \phi$ ' means that every model of $\Gamma$ is a model of $\phi$. The two notions can be related by soundness and completeness.
3. Explain the axiom of extensionality. Let $R$ be the set of numbers $\{1,2,3,4\}$, let $S$ be the set $\{3,4,5,6\}$ and let $T$ be $\{5,6,7,8\}$. Write down the elements of the following sets: [20]

Extensionality: $\forall X \forall Y(X=Y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$
(i) $R \cup T=\{1,2,3,4,5,6,7,8\}$
(ii) $R \cap(S \cup T)=\{3,4\}$
(iii) $\mathcal{P}(R \cap S)=\{\{3\},\{4\},\{3,4\}, \phi\}$
(iv) $\mathcal{P}(R) \cap \mathcal{P}(S)=\{\{3\},\{4\},\{3,4\}, \phi$
(v) The set $y$ of all elements of $S$ that exceed some element of $S-T$ has no members.
(b) Define reflexivity, symmetry and transitivity. For each of the following relations, say which of these properties it possesses (on the domain of presently living people):

A relation, $R$, is reflexive with respect to a given domain if, and only if, $\forall x R x x$.
A relation, $R$, is symmetric with respect to a given domain if, and only if, $\forall x \forall y(R x y \rightarrow$ Ryx).
A relation, $R$, is transitive with respect to a given domain if, and only if, $\forall x \forall y \forall z((R x y \wedge$ $R y z) \rightarrow R x z)$.
(i) $x$ is at least one inch taller than $y$ not-R, not-S, T
(ii) $x$ is at most one inch taller than $y$

R, not-S, not-T
(iii) $x$ is taller than $y$ iff $y$ is taller than $x$

R, S, T
(iv) $x$ is exactly 80 years older than $y$
not-R, not-S, T
(v) $x$ is a brother of $y$
not-R, not-S, not-T
(vi) $x$ and $y$ are brothers
not-R, S, not-T
(vii) $x$ and $y$ have the same first name or the same last name

R, S, not-T
(viii) $x$ owns a helicopter iff $y$ does

R, not-S, T
(ix) $x$ is David Hasselhoff and $y$ is not
not-R, not-S, T
(x) Most people prefer $x$ to $y$
not-R, not-S, not-T
4. (a) Write down the probability axioms and the definition of conditional probability. [10]

Probability Axioms:
$\operatorname{Pr}(V)=1$, where V is our sample space (set of possible outcomes).
$\operatorname{Pr}(X) \geq 0$ where $X \in \mathcal{P}(V)$
if $X \cap Y=\emptyset$ then $\operatorname{Pr}(X \cup Y)=\operatorname{Pr}(X)+\operatorname{Pr}(Y)$
$\operatorname{Pr}(A \mid B)=\frac{\operatorname{Pr}(A \cap B)}{\operatorname{Pr}(B)}$
(b) I draw three cards at random without replacement from a standard pack of cards. What is the probability:
(i) That the first two are aces? [5]

$$
\operatorname{Pr}(F 2 A)=\frac{1}{13}+\frac{3}{51}=\frac{1}{221}
$$

(ii) That the first two are aces given that the first is an ace? [10]

$$
\operatorname{Pr}(F 2 A \mid F A)=\frac{\operatorname{Pr}(F 2 A) \cap \operatorname{Pr}(F A)}{\operatorname{Pr}(F A)}=\frac{\operatorname{Pr}(F 2 A)}{\operatorname{Pr}(F A)}=\frac{\frac{1}{211}}{\frac{1}{13}}=\frac{1}{17}
$$

(iii) That the first two are aces given that the third is the King of Hearts? [15]

$$
\operatorname{Pr}(F 2 A \mid 3 K H)=\frac{\operatorname{Pr}(F 2 A \cap 3 K H)}{\operatorname{Pr}(3 K H)}=\frac{\frac{1}{\frac{2}{21} 1} \frac{1}{50}}{\frac{51}{52} x \frac{50}{51} x \frac{1}{50}}=\frac{2}{425}
$$

(c) I have 2 bags. One of them contains 50 red balls and 50 black balls. The other contains 60 red balls and 40 black balls. I cannot tell which is which. I choose a bag at random and draw a ball from it (without putting it back). It is red. Then I draw another ball from the same bag. What is the probability that this second ball is red? [30]
$\operatorname{Pr}(2 R \mid 1 R)=\frac{\operatorname{Pr}(2 R \cap 1 R)}{\operatorname{Pr}(1 R)}$
$\operatorname{Pr}(2 R \cap 1 R)=\left(\frac{1}{2} x \frac{1}{2} x \frac{49}{99}\right)+\left(\frac{1}{2} x \frac{3}{5} x \frac{59}{99}\right)=\frac{599}{1980}$
$\operatorname{Pr}(1 R)=\left(\frac{1}{2} x \frac{1}{2}\right)+\left(\frac{1}{2} x \frac{3}{5}\right)=\frac{11}{20}$
$\operatorname{Pr}(2 R \mid 1 R)=\frac{\frac{599}{1900}}{\frac{10}{20}}=\frac{599}{1089}$
(d) Four balls are placed in a bowl. One is green, one is black and the other two are yellow. The bowl is shaken and someone draws two balls from the bowl. He looks at the two balls and announces that at least one of them is yellow. What is the probability that the other ball he has drawn out is also yellow? [30]
$\operatorname{Pr}(2 Y \mid 1 Y)=\frac{\operatorname{Pr}(2 Y) \cap \operatorname{Pr}(1 Y)}{\operatorname{Pr}(1 Y)}$
$\operatorname{Pr}(1 Y)=\left(\frac{1}{2} x \frac{2}{3}\right)+\left(\frac{1}{2} x \frac{2}{3}\right)+\left(\frac{1}{2} x \frac{1}{3}\right)=\frac{5}{6}$
$\operatorname{Pr}(2 Y \cap 1 Y)=\frac{1}{2} x \frac{1}{3}=\frac{1}{6}$
$\operatorname{Pr}(2 Y \mid 1 Y)=\frac{\frac{1}{6}}{6}=\frac{1}{5}$

