Model answers are in blue. In some cases, multiple answers are acceptable. Any student who thinks they have spotted a mistake in a model answer should email tecb2 *at* cam *dot* ac *dot* uk

SECTION A

Answer **all** questions in section A.

(1) Could there be:

(a) a valid argument with a true conclusion but a false premise?

Yes. For example:

Socrates is a man and a carrot.

So: Socrates is a man.

(b) a valid argument with only false premises and a false conclusion?

Yes. For example:

Socrates is a man and a carrot.

So: Socrates is a carrot.

(c) a sound argument whose conclusion is a tautology?

Yes. For example:

It is raining.

So: Either it is raining or it is not raining.

(d) a sound argument with a contradiction as a premise?

No. By definition, an argument is sound iff it is valid and all its premises are true. So, in particular, every premise of a sound argument is true. And no contradiction is true.

If so, provide an example of such an argument. If not, explain why not.

(2) Use truth-tables (complete or partial) to assess the following:

(a) $A \lor B, B \lor C, \neg A \vDash B \land C$

This claim is false, as the following partial truth table shows.

A	B	C	$A \lor B$	$B \lor C$	$\neg A$	$B \land C$
F	T	F	T	T	T	F

(b) $(\neg A \leftrightarrow B) \vDash \neg(\neg A \leftrightarrow \neg B)$

This claim is true, as the following complete truth table shows.

A	B	$(\neg A$	$\leftrightarrow$	$B)$	$\neg($	$\neg A$	$\leftrightarrow$	$\neg B)$
T	T	f	F		F	f	t	f
T	F	f	T		T	f	f	t
F	T	t	T		T	t	f	f
F	F	t	F		F	t	t	t

(c) $\vDash (A \to B) \lor (B \to A)$

This claim is true, as the following complete truth table shows.

A	B	$(A \to B)$	$\lor$	$(B \to A)$
T	T	t	T	t
T	F	f	T	t
F	T	t	T	f
F	F	t	T	t

(3) Using the formal proof system from *forallx*, show that:
$\forall x(Fx \to Gx), \exists x(Fx \wedge Hx) \vdash \exists x(Gx \wedge Hx)$

1	$\forall x(Fx \to Gx)$	
2	$\exists x(Fx \wedge Hx)$	
3	$\quad Fa \wedge Ha$	
4	$\quad Fa$	$\wedge$E 3
5	$\quad Ha$	$\wedge$E 3
6	$\quad Fa \to Ga$	$\forall$E 1
7	$\quad Ga$	$\to$E 6, 4
8	$\quad Ga \wedge Ha$	$\wedge$I 7, 5
9	$\quad \exists x(Gx \wedge Hx)$	$\exists$I 8
10	$\exists x(Gx \wedge Hx)$	$\exists$E 2, 3–9

(4) Provide examples of relations with the following properties:

(a) reflexive and symmetric but not transitive
The relation, on the domain of people, given by:
x and *y* are the same height or differ in height by no more than 10cm.

(b) transitive and symmetric but not reflexive
The empty relation (on any domain).

(c) reflexive but neither symmetric nor transitive
The relation on the numbers 1, 2, and 3 whose extension is:
$\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 1,2\rangle, \langle 2,3\rangle, \langle 3,1\rangle$.

(5) You roll two fair six-sided dice, once. Calculate the probability that:

(a) you roll 11.

$$\frac{2}{36} = \frac{1}{18}$$

(b) you roll 11, given that at least one of the dice showed a 6.

$$\frac{2}{11}$$

(c) you roll 11, given that both dice show the same number.

$$0$$

SECTION B

Answer any **two** questions from section B.

(6) Using the following symbolisation key:

Domain: people
D: $\_\_\_\__1$ is a drummer
B: $\_\_\_\__1$ is a bassist
L: $\_\_\_\__1$ likes $\_\_\_\__2$
a: Ali
b: Barker

symbolise all of the following English sentences as best you can in FOL:

(a) Ali likes Barker, and also other people.
$Lab \land \exists x(\neg x = b \land Lax)$

(b) Every bassist likes a drummer.
$\forall x(Bx \to \exists y(Dy \land Lxy))$.
Note: the original English sentence is potentially ambiguous; I have given it a reading which allows (for example) that different bassists may like different drummers.

(c) The drummer who likes Ali is not Barker.
$\exists x(Dx \land Lxa \land \forall y((Dy \land Lya) \to x = y) \land \neg x = b)$

(d) Provided Ali likes Barker, some bassist likes some drummer.
$Lab \to \exists x\exists y(Bx \land Dy \land Lxy)$

(e) Exactly two drummers other than Barker like Ali.
$\exists x\exists y(\neg x = y \land \neg x = b \land \neg y = b \land Dx \land Dy \land Lxa \land Lya \land$
$\quad \forall z[(Dz \land Lza) \to (z = x \lor z = y)])$
Note: the original English sentence may be intended to imply that Barker is a drummer who likes Ali. If so, then we should offer:
$Db \land Lba \land \exists x\exists y(\neg x = y \land \neg x = b \land \neg y = b \land Dx \land Dy \land Lxa \land Lya \land$
$\quad \forall z[(Dz \land Lza) \to (z = x \lor z = y \lor z = b)])$

(f) The drummer who likes Ali is not the bassist who likes Barker.
$\exists x\exists y(Dx \land Lxa \land \forall z((Dz \land Lza) \to x = z) \land$
$\quad By \land Lyb \land \forall z((Bz \land Lzb) \to y = z) \land x \neq y)$

(g) Barker likes each of the three bassists.
$\exists x\exists y\exists z(Bx \land By \land Bz \land \neg x = y \land \neg y = z \land \neg x = z \land$
$\quad \forall w[Bw \to (x = w \lor y = w \lor z = w)] \land Lbx \land Lby \land Lbz)$

(h) For every drummer who likes a bassist, some other drummer likes no one but Ali.
$\forall x([Dx \land \exists y(By \land Lxy)] \to \exists z[Dz \land \neg z = x \land \forall y(Lzy \to y = a)])$
Note: the original English sentence may be intended so that its last part is read '…likes Ali but no one other than Ali'. If so, the conditional in my answer, '$Lzy \to y = a$', should be changed to a biconditional, '$Lzy \leftrightarrow y = a$'

(i) Someone who is liked by nobody likes everyone.
$\exists x(\forall y\neg Lyx \land \forall zLxz)$
Note: the original English sentence may be intended to be read e.g. '…likes everyone *other than themselves*'. If so, the answer should be $\exists x(\forall y\neg Lyx \land \forall z(\neg z = x \to Lxz))$

(j) Someone likes all and only those who do not like themselves.
$\exists x\forall y(Lxy \leftrightarrow \neg Lyy)$.

(7) Grange Knoll is a school with a population of 800 children and 200 adults. Sadly, 40 children and 40 adults in Grange Knoll have the flu. A member of the Grange Knoll is chosen at random; calculate the probability that:
In all answers, let C be the event of being/sampling a child and F be the event of being/sampling someone with flu. (Note that $\overline{C}$ is the event of being/sampling an adult.)

(a) they have the flu.

$$\Pr(F) = \frac{40 + 40}{800 + 200} = \frac{2}{25}$$

(b) they have the flu, given that they are a child.

$$\Pr(F \mid C) = \frac{40}{800} = \frac{1}{20}$$

(c) they have the flu, given that they are an adult.

$$\Pr(F \mid \overline{C}) = \frac{40}{200} = \frac{1}{5}$$

(d) they are a child, given that they have the flu.

$$\Pr(C \mid F) = \frac{40}{80} = \frac{1}{2}$$

(e) they are an adult, given that they do not have the flu.

$$\Pr(\overline{C} \mid \overline{F}) = \frac{160}{920} = \frac{4}{23}$$

A test has been developed, to determine whether or not someone has the flu. Among those who have the flu, the test delivers a positive verdict 95% of the time. Among those who do not have the flu, the test delivers a positive verdict 5% of the time. A member of Grange Knoll is chosen at random; calculate the probability that:
In all subsequent answers, let V be the event of the test returning a positive verdict.

(f) they have the flu, given that the test delivered a positive verdict

$$\begin{aligned}
\Pr(F \mid V) &= \frac{\Pr(F \wedge V)}{\Pr(V)} \\
&= \frac{\Pr(F \wedge V)}{\Pr(F \wedge V) + \Pr(\overline{F} \wedge V)} \\
&= \frac{\frac{2}{25} \times \frac{19}{20}}{\frac{2}{25} \times \frac{19}{20} + \frac{23}{25} \times \frac{1}{20}}, \text{ using the answer to (a)} \\
&= \frac{2 \times 19}{2 \times 19 + 23 \times 1} \\
&= \frac{38}{61}
\end{aligned}$$

(g) they have the flu, given both that they are a child and that the test delivered a positive verdict

$$\begin{aligned}
\Pr(F \mid (C \wedge V)) &= \frac{\Pr(F \wedge C \wedge V)}{\Pr(C \wedge V)} \\
&= \frac{\Pr(F \wedge C \wedge V)}{\Pr(F \wedge C \wedge V) + \Pr(\overline{F} \wedge C \wedge V)} \\
&= \frac{\frac{40}{1000} \times \frac{19}{20}}{\frac{40}{1000} \times \frac{19}{20} + \frac{760}{1000} \times \frac{1}{20}} \\
&= \frac{40 \times 19}{40 \times 19 + 760 \times 1} \\
&= \frac{1}{2}
\end{aligned}$$

(h) they are an adult, given that the test delivered a positive verdict

$$\begin{aligned}
\Pr(\overline{C} \mid V) &= \frac{\Pr(\overline{C} \wedge V)}{\Pr(V)} \\
&= \frac{\Pr(\overline{C} \wedge F \wedge V) + \Pr(\overline{C} \wedge \overline{F} \wedge V)}{\Pr(V \wedge F) + \Pr(V \wedge \overline{F})} \\
&= \frac{\frac{40}{1000} \times \frac{19}{20} + \frac{160}{1000} \times \frac{1}{20}}{\frac{19}{20} \times \frac{2}{25} + \frac{1}{20} \times \frac{23}{25}}, \text{ calculating denominator as in (f)} \\
&= \frac{1 \times 19 + 4 \times 1}{19 \times 2 + 1 \times 23} \\
&= \frac{23}{61}
\end{aligned}$$

(8) Using the formal proof system from *forallx*, show each of the following:

(a) $\forall x \exists y (Rxy \vee Ryx), \forall x \neg Rmx \vdash \exists x Rxm$

1	$\forall x \exists y (Rxy \vee Ryx)$	
2	$\forall x \neg Rmx$	
3	$\exists y (Rmy \vee Rym)$	∀E 1
4	$Rma \vee Ram$	
5	$\neg Rma$	∀E 2
6	Ram	DS 4, 5
7	$\exists x Rxm$	∃I 6
8	$\exists x Rxm$	∃E 3, 4–7

(b) $\forall x (\exists y Lxy \rightarrow \forall z Lzx), Lab \vdash \forall x Lxx$

1	$\forall x (\exists y Lxy \rightarrow \forall z Lzx)$	
2	Lab	
3	$\exists y Lay \rightarrow \forall z Lza$	∀E 1
4	$\exists y Lay$	∃I 2
5	$\forall z Lza$	→E 3, 4
6	Lca	∀E 5
7	$\exists y Lcy$	∃I 6
8	$\exists y Lcy \rightarrow \forall z Lzc$	∀E 1
9	$\forall z Lzc$	→E 8, 7
10	Lcc	∀E 9
11	$\forall x Lxx$	∀I 10

(c) $\forall x((Px \wedge \exists y Lyx) \to Dx), \forall x(Dx \to \neg \exists y Lyx) \vdash \forall x(Px \to \neg \exists y Lyx)$

1	$\forall x((Px \wedge \exists y Lyx) \to Dx)$	
2	$\forall x(Dx \to \neg \exists y Lyx)$	
3	$\quad Pa$	
4	$\quad\quad \exists y Lya$	
5	$\quad\quad Pa \wedge \exists y Lya$	$\to$I 3, 4
6	$\quad\quad (Pa \wedge \exists y Lya) \to Da$	$\forall$E 1
7	$\quad\quad Da$	$\to$E 6, 5
8	$\quad\quad Da \to \neg \exists y Lya$	$\forall$E 2
9	$\quad\quad \neg \exists y Lya$	$\to$E 8, 7
10	$\quad\quad \bot$	$\neg$E 4, 9
11	$\quad \neg \exists y Lya$	$\neg$I 4–10
12	$Pa \to \neg \exists y Lya$	$\to$I 3–11
13	$\forall x(Px \to \neg \exists y Lyx)$	$\forall$I 12

(d) $\forall x(Lax \to \forall y(Lay \to x = y)), \neg Pc \vdash Lac \to \forall x(Lax \to \neg Px)$

1	$\forall x(Lax \to \forall y(Lay \to x = y))$	
2	$\neg Pc$	
3	$Lac \to \forall y(Lay \to c = y)$	$\forall$E 1
4	$\quad Lac$	
5	$\quad \forall y(Lay \to c = y)$	$\to$E 3, 4
6	$\quad\quad Lab$	
7	$\quad\quad Lab \to c = b$	$\forall$E 5
8	$\quad\quad c = b$	$\to$E 7, 6
9	$\quad\quad \neg Pb$	=E 8, 2
10	$\quad Lab \to \neg Pb$	$\to$I 6–9
11	$\quad \forall x(Lax \to \neg Px$	$\forall$I 10
12	$Lac \to \forall x(Lax \to \neg Px)$	$\to$I 4–11

(e) $\forall x(\neg Mx \vee Lax), \forall x(Cx \to Lax), \forall x(Mx \vee Cx) \vdash \forall x Lax$

1	$\forall x(\neg Mx \vee Lax)$	
2	$\forall x(Cx \to Lax)$	
3	$\forall x(Mx \vee Cx)$	
4	$\neg Mc \vee Lac$	$\forall$E 1
5	$Cc \to Lac$	$\forall$E 2
6	$Mc \vee Cc$	$\forall$E 3
7	$\quad \neg Lac$	
8	$\quad \neg Mc$	DS 4, 7
9	$\quad Cc$	DS 6, 8
10	$\quad Lac$	$\to$E 5, 9
11	$\quad \bot$	$\neg$E 7, 10
12	$\neg\neg Lac$	$\neg$I 7–11
13	Lac	DNE 12

(f) $\forall x(Lax \to x = a), \forall x(\exists y Lxy \to x = a) \vdash \forall x(\exists y Lyx \to x = a)$

1	$\forall x(Lax \to x = a)$	
2	$\forall x(\exists y Lxy \to x = a)$	
3	$\quad \exists y Lyd$	
4	$\quad\quad Led$	
5	$\quad\quad \exists y Ley$	$\exists$I 4
6	$\quad\quad \exists y Ley \to e = a$	$\forall$E 2
7	$\quad\quad e = a$	$\to$E 6, 5
8	$\quad\quad Lad$	$=$E 7, 4
9	$\quad Lad$	$\exists$E 3, 4–8
10	$\quad Lad \to d = a$	$\forall$E 1
11	$\quad d = a$	$\to$E 10, 9
12	$\exists y Lyd \to d = a$	$\to$I 3–11
13	$\forall x(\exists y Lyx \to x = a)$	$\forall$I 12

(9) Attempt all parts of this question

(a) Let $A = \{\text{Algeria, Benin, Chad}\}$, $B = \{\text{Benin, Chad}\}$, $C = \{\text{Chad, Djibouti, Egypt}\}$, and $D = \{\text{Fiji}\}$. Calculate the members of each of the following sets:

(i) $(B - C) \cup D$

$$\begin{aligned}(B - C) \cup D &= \{\text{Benin}\} \cup \{\text{Fiji}\} \\ &= \{\text{Benin, Fiji}\}\end{aligned}$$

(ii) $(A \cap B) \cup (C \cap D)$

$$\begin{aligned}(A \cap B) \cup (C \cap D) &= \{\text{Benin, Chad}\} \cup \varnothing \\ &= \{\text{Benin, Chad}\}\end{aligned}$$

(iii) $A \times B$

$$\begin{aligned}A \times B = \{&\langle\text{Algeria, Benin}\rangle, \langle\text{Algeria, Chad}\rangle, \langle\text{Benin, Benin}\rangle, \langle\text{Benin, Chad}\rangle, \\ &\langle\text{Chad, Benin}\rangle, \langle\text{Chad, Chad}\rangle\}\end{aligned}$$

(iv) $\{x : x \subseteq A \cap B)$

$$\begin{aligned}\{x : x \subseteq A \cap B\} &= \{x : x \subseteq \{\text{Benin, Chad}\}\} \\ &= \{\varnothing, \{\text{Benin}\}, \{\text{Chad}\}, \{\text{Benin, Chad}\}\}\end{aligned}$$

(v) $\wp(B \cap C)$

$$\begin{aligned}\wp(B \cap C) &= \wp(\{\text{Chad}\}) \\ &= \{\varnothing, \{\text{Chad}\}\}\end{aligned}$$

(vi) $\wp(\wp(C \cap D))$

$$\begin{aligned}\wp(\wp(C \cap D)) &= \wp(\wp(\varnothing)) \\ &= \wp(\{\varnothing\}) \\ &= \{\varnothing, \{\varnothing\}\}\end{aligned}$$

(vii) $(A - C) \times D$

$$\begin{aligned}(A - C) \times D &= \{\text{Algeria, Benin}\} \times \{\text{Fiji}\} \\ &= \{\langle\text{Algeria, Fiji}\rangle, \langle\text{Benin, Fiji}\rangle\}\end{aligned}$$

(viii) $\wp(B - C) \times \wp(D)$

$$\begin{aligned}\wp(B - C) \times \wp(D) &= \wp(\{\text{Benin}\}) \times \wp(\{\text{Fiji}\}) \\ &= \{\varnothing, \{\text{Benin}\}\} \times \{\varnothing, \{\text{Fiji}\}\} \\ &= \{\langle\varnothing, \varnothing\rangle, \langle\varnothing, \{\text{Fiji}\}\rangle, \langle\{\text{Benin}\}, \varnothing\rangle, \langle\{\text{Benin}\}, \{\text{Fiji}\}\rangle\}\end{aligned}$$

(b) Is there any set A such that $\wp(A) = \varnothing$? If so, give an example; if not, explain why not.
There is no such set. After all, $\varnothing \subseteq A$ and hence $\varnothing \in \wp(A)$, for any set A.

(c) Show that $A - (C - A) = A$, no matter what sets A and C are.
First, suppose that $x \in A - (C - A)$, i.e. $x \in A$ but $x \notin (C - A)$. So in particular, $x \in A$. Generalising on x, this shows that $A - (C - A) \subseteq A$.
Next, suppose that $x \in A$. Then $x \notin C - A$. So $x \in A - (C - A)$. Generalising on x, this shows that $A \subseteq A - (C - A)$.
By Extensionality, it follows that $A = A - (C - A)$.

END OF PAPER