SECTION A
Answer **all** questions in section A.

(1) Which of the following claims are true, and which are false? Briefly explain your answers:
1 mark for the right answer; 2 marks for explanation; no partial credit

(a) Every invalid argument can be turned into a valid argument by adding a premise.
True – just add the conclusion as a premise

(b) Some valid argument can be turned into an invalid argument by adding a premise.
False – an argument is valid iff whenever all premises are true the conclusion is too. Adding more premises only makes the initial condition harder to satisfy.

(c) Every sound argument can be turned into an unsound argument by adding a premise.
True – just add a false premise.

(2) Consider an interpretation, whose domain is just the numbers 1, 2, and 3, and where the predicate '*R*' is to be true of, and only of:

$$\langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 2,3\rangle$$

Given this interpretation, state the truth values of each of these sentences (you do not need to explain your answers):
1 mark for (a), 2 for each of (b)–(e)

(a) $\exists x Rxx$ True
(b) $\exists x \forall y Rxy$ True
(c) $\forall x \exists y Rxy$ False
(d) $\forall x \exists y Ryx$ True
(e) $\exists x \exists y (Rxy \wedge \neg Ryx)$ True

(3) Construct complete truth tables for each of these three sentences:

$$\neg(A \to B), \neg A \to B, A \to \neg B$$

1 mark for each of these 3 tables:

A	B	$\neg(A \to B)$	$\neg A \to B$	$A \to \neg B$
T	T	F	T	F
T	F	T	T	T
F	T	F	T	T
F	F	F	F	T

Use these truth tables to determine whether each of the following claims is true or false:
1 mark for each correct answer; carrying forward errors on the truth tables if necessary

(a) $\neg(A \to B) \vDash \neg A \to B$ True
(b) $\neg(A \to B) \vDash A \to \neg B$ True
(c) $\neg A \to B \vDash \neg(A \to B)$ False
(d) $\neg A \to B \vDash A \to \neg B$ False
(e) $A \to \neg B \vDash \neg(A \to B)$ False
(f) $A \to \neg B \vDash \neg A \to B$ False

(4) For each of the following relations on the domain of countries, say whether they are reflexive, whether they are symmetric, and whether they are transitive: 1 mark for each property

(a) x has exactly the same area as y
ref, sym, trans

(b) x has at least as many inhabitants as y
ref, not sym, trans

(c) x has a land border with y
not ref, sym, not trans

(5) Two cards are drawn at random, without replacement, from a standard deck of cards (i.e. 4 of the 52 cards are aces). What is the probability that: 3 marks per question, partial credit iff there's e.g. a very simple error in calculation e.g. mistranscribing what's in the calculator

(a) both are aces?

$$\frac{4}{52} \times \frac{3}{51} = \frac{1}{221}$$

(b) exactly one is an ace?

$$\begin{aligned} \frac{4}{52} \times \frac{48}{51} + \frac{48}{52} \times \frac{4}{51} &= \frac{8 \times 48}{52 \times 51} \\ &= \frac{32}{221} \end{aligned}$$

(c) at least one is an ace?

$$\frac{33}{221}$$

SECTION B

Answer any **two** questions from section B.

(6) Attempt both parts of this question. Throughout, use this symbolisation key:

domain: all physical objects

C: $\_\_\_\__1$ is a cup

S: $\_\_\_\__1$ is a saucer

A: $\_\_\_\__1$ belongs to Anna

O: $\_\_\_\__1$ is on top of $\_\_\_\__2$

(a) Offer natural-language renditions of the following sentences of FOL. Make the natural-language sentences as natural as possible, whilst keeping the same truth-conditions as the FOL sentences:

(i) $\forall x(Ax \rightarrow \neg(Cx \vee Sx))$

Anna owns no cups nor saucers – 1 mark, 'unnatural' answers get 0.5

(ii) $\forall x((Cx \wedge \exists y(Sy \wedge Oxy)) \rightarrow Ax)$

Every cup on a saucer belongs to Anna – 1 mark, 'unnatural' answers get 0.5

(iii) $\exists x(Cx \wedge Ax \wedge \forall y((Cy \rightarrow Ay) \rightarrow x = y))$

Anna owns a cup, and everything is a cup

2 marks on offer here; but NB it's tricksy! 0 marks for people who read it as 'Anna owns exactly one cup'.

(iv) $\exists x(Cx \wedge \forall y(Cy \rightarrow x = y) \wedge Ax)$

The cup is Anna's – 1 mark

(v) $\exists x(\forall y(Sx \leftrightarrow x = y) \wedge \forall y((Cy \wedge Ay) \rightarrow \neg Oyx))$

None of Anna's cups are on the saucer – 2 marks. 0 marks for people who mix up the uniqueness of the saucer with that of the cup, etc.

(b) Symbolise all of the following English sentences as best you can in FOL. Comment on any difficulties you encounter:

(i) Anna owns a cup but no saucer

$\exists x(Cx \wedge Ax) \wedge \neg\exists y(Sy \wedge Ax)$ – 1 mark

(ii) All of Anna's cups are on saucers

$\forall x((Ax \wedge Cx) \rightarrow \exists y(Sy \wedge Oxy))$ – 2 marks

(iii) Anna's cup is on a saucer

$\exists x(Ax \wedge Cx \wedge \forall y((Ay \wedge Cy) \rightarrow x = y) \wedge \exists z(Sz \wedge Oxz))$ – 2 marks

(iv) Anna's saucer has exactly two things on it

$\exists x(Sx \wedge Ax \wedge \forall y((Sy \wedge Ay) \rightarrow x = y) \wedge \exists s \exists t(\neg s = t \wedge Osx \wedge Otx \wedge \forall z(Ozx \rightarrow (z = x \vee z = y)))$ – 4 marks. Max of 2 marks if you miss out the exactness condition, or that Anna's saucer is a definite description.

(v) The cup on Anna's saucer indeed belongs to Anna

$\exists x(Sx \wedge Ax \wedge \forall y((Sy \wedge Ay) \rightarrow x = y) \wedge \exists s(Cs \wedge Osx \wedge \forall y((Cy \wedge Oyx) \rightarrow s = x) \wedge As))$ – 4 marks. NB that there are two definite descriptions here.

(7) Using the formal proof system from *forallx:Cambridge*, show each of the following:

(a): 3 marks; (b): 4 marks; (c): 6 marks; (d): 7 marks

(a) $\exists xFx \rightarrow \forall yGy \vdash \forall x\forall y(Fx \rightarrow Gy)$

1	$\exists x Fx \rightarrow \forall y Gy$	
2	$\quad Fa$	
3	$\quad \exists x Fx$	∃I 2
4	$\quad \forall y Gy$	→E 1, 3
5	$\quad Gb$	∀E 4
6	$Fa \rightarrow Gb$	→I 2–5
7	$\forall y(Fa \rightarrow Gy)$	∃I 6
8	$\forall x \forall y(Fx \rightarrow Gy)$	∀I 7

(b) $\neg\exists x(Fx \wedge Gx) \vdash \forall x(Gx \rightarrow \neg Fx)$

1	$\neg\exists x(Fx \wedge Gx)$	
2	$\quad Ga$	
3	$\quad\quad Fa$	
4	$\quad\quad Fa \wedge Ga$	∧I 3, 2
5	$\quad\quad \exists x(Fx \wedge Gx)$	∃I 4
6	$\quad\quad \bot$	¬E 1, 5
7	$\quad \neg Fa$	¬I 3–6
8	$Ga \rightarrow \neg Fa$	→I 2–7
9	$\forall x(Gx \rightarrow \neg Fx)$	∀I 8

(c) $\forall x \exists y Rxy, \forall x \neg Rxx \vdash \exists x \exists y \neg x = y$

1	$\forall x \exists y Rxy$	
2	$\forall x \neg Rxx$	
3	$\exists y Ray$	∀E 1
4	$\neg Raa$	∀E 2
5	$\quad Rab$	
6	$\quad\quad a = b$	
7	$\quad\quad Raa$	=E 6, 5
8	$\quad\quad \bot$	¬I 7, 4
9	$\quad \neg a = b$	¬I 6–8
10	$\quad \exists y \neg a = y$	∃I 9
11	$\quad \exists x \exists y \neg x = y$	∃I 10
12	$\exists x \exists y \neg x = y$	∃E 3, 5–11

(d) $Fb, Lab, \exists x \forall y(Lay \rightarrow x = y) \vdash \forall x(Lax \rightarrow Fx)$

1	Fb	
2	Lab	
3	$\exists x \forall y(Lay \to x = y)$	
4	$\forall y(Lay \to e = y)$	
5	$Lab \to e = b$	∀E 4
6	$e = b$	→E 5, 2
7	Fe	=E 6, 1
8	Lac	
9	$Lac \to e = c$	∀E 4
10	$e = c$	→E 9, 8
11	Fc	=E 10, 7
12	$Lac \to Fc$	→I 8–11
13	$\forall x(Lax \to Fx)$	∀E 12
14	$\forall x(Lax \to Fx)$	∃E 3, 4–13

(8) Answer both parts of this question.

(a) Jelly beans are manufactured in two colours: red and green. They are manufactured so that 5% of the beans of one colour are bitter, and 10% of the beans of the other colour are bitter; but you don't know which is which. What probability should you assign to the hypothesis that 10% of red beans are bitter, in each of the following distinct scenarios: (ai): 3 marks; (aii): 4 marks; (aiii): 5 marks; (bi): 1 mark; (bii): 7 marks; partial credit is pretty much only available for simple calculator error.

(i) You eat a single red bean, and it is bitter.

$$\begin{aligned} Pr(T \mid B) &= \frac{Pr(T \cap B)}{Pr(T \cap B) + Pr(\overline{T} \cap B)} \\ &= \frac{0.5 \times 0.1}{0.5 \times 0.1 + 0.5 \times 0.05} \\ &= \frac{2}{3} \end{aligned}$$

(ii) You eat a single red bean, and it is not bitter.

$$\begin{aligned} Pr(T \mid \overline{B}) &= \frac{Pr(T \cap \overline{B})}{Pr(T \cap \overline{B}) + Pr(\overline{T} \cap \overline{B})} \\ &= \frac{0.5 \times 0.9}{0.5 \times 0.9 + 0.5 \times 0.95} \\ &= \frac{18}{37} \approx 0.486 \end{aligned}$$

(iii) You eat two red beans, and neither is bitter.

$$
\begin{aligned}
Pr(T \mid \overline{B_1} \cap \overline{B_2}) &= \frac{Pr(T \cap \overline{B_1} \cap \overline{B_2})}{Pr(T \cap \overline{B_1} \cap \overline{B_2}) + Pr(\overline{T} \cap \overline{B_1} \cap \overline{B_2})} \\
&= \frac{0.5 \times 0.9 \times 0.9}{0.5 \times 0.9 \times 0.9 + 0.5 \times 0.95 \times 0.95} \\
&= \frac{8100}{8100 + 9025} \\
&= \frac{324}{685} \approx 0.473
\end{aligned}
$$

(b) A D20 is a fair, twenty-sided die, whose faces are numbered 1 through 20.

(i) You roll two D20. What is the probability that they both show the same number?

$$\frac{1}{20}$$

(ii) You roll eight D20. What is the probability that at least two of the eight dice show the same number?

$$
\begin{aligned}
1 - \left(\frac{19}{20} \times \frac{18}{20} \times \ldots \times \frac{13}{20}\right) &= \frac{20^7 - 253,955,520}{20^7} \\
&= \frac{1,026,044,480}{1,280,000,000} \\
&= \frac{3,206,389}{4,000,000} = 0.80159725
\end{aligned}
$$

(6) Answer all parts of this question.

(a) Let L be the set of all labradors. Let B be the set of all bulldogs. (Note that no labrador is a bulldog, and no bulldog is a labrador.) Using the following symbolisation key:

T: $\_\_\_\__1$ has a longer tail than $\_\_\_\__2$

f: Fido (who is a labrador)

write down set-theoretic expressions for the following sets:

2 marks for (a) overall.

(i) The set of all bulldogs with tails longer than Fido's.

$\{x : x \in B \wedge Txf\}$

(ii) The set of all bulldogs with tails longer than every labrador's.

$\{x : x \in B \wedge \forall y(Ly \to Txy)\}$

(iii) The set of ordered pairs, one of which is a labrador and the other of which is a bulldog.

$L \times B \cup B \times L$

(iv) The set of ordered pairs, one of which is a labrador other than Fido and the other of which is a bulldog.

$(L \setminus \{f\}) \times B \cup B \times (L \setminus \{f\})$

(b) Let $A = \{0, 2, 4, 6\}$, $B = \{0, 1, 2, \{4\}, \{6\}\}$, $C = \{\{0\}\}$. Calculate the members of the following sets:

(i) $(A \cup B) \cap C$
it's empty: 1 mark.
Candidates can say "it's empty", or write e.g. $(A \cup B) \cap C = \varnothing$. That is: they can either write down a term for the set, or just list its members. But if they mix between these without being clear, in part (b), they lost marks.

(ii) $B \times C$
$\langle 0, \{0\}\rangle$,
$\langle 1, \{0\}\rangle$,
$\langle 2, \{0\}\rangle$,
$\langle \{4\}, \{0\}\rangle$,
$\langle \{6\}, \{0\}\rangle$: 1 mark

(iii) $\wp(A) - \wp(B)$
$\{4\}$,
$\{6\}$,
$\{0, 4\}$,
$\{0, 6\}$,
$\{2, 4\}$,
$\{2, 6\}$,
$\{4, 6\}$,
$\{0, 2, 4\}$,
$\{0, 2, 6\}$,
$\{0, 4, 6\}$,
$\{2, 4, 6\}$,
$\{0, 2, 4, 6\}$: 3 marks

(iv) $\wp(C) \times \wp(B - A)$
$\langle \varnothing, \varnothing\rangle$,
$\langle \varnothing, \{1\}\rangle$,
$\langle \varnothing, \{\{4\}\}\rangle$,
$\langle \varnothing, \{\{6\}\}\rangle$,
$\langle \varnothing, \{1, \{4\}\}\rangle$,
$\langle \varnothing, \{1, \{6\}\}\rangle$,
$\langle \varnothing, \{\{4\}, \{6\}\}\rangle$,
$\langle \varnothing, \{1, \{4\}, \{6\}\}\rangle$,
$\langle \{\{0\}\}, \varnothing\rangle$,
$\langle \{\{0\}\}, \{1\}\rangle$,
$\langle \{\{0\}\}, \{\{4\}\}\rangle$,
$\langle \{\{0\}\}, \{\{6\}\}\rangle$,
$\langle \{\{0\}\}, \{1, \{4\}\}\rangle$,
$\langle \{\{0\}\}, \{1, \{6\}\}\rangle$,
$\langle \{\{0\}\}, \{\{4\}, \{6\}\}\rangle$,
$\langle \{\{0\}\}, \{1, \{4\}, \{6\}\}\rangle$: 5 marks
Some partial credit available if they miscalculate e.g. $\wp(B - A)$ but then get the right answer modulo that mistake

(c) What are the members of $\wp(\wp(\varnothing))$?
$\varnothing$ and $\{\varnothing\}$: 4 marks

(d) Show that, if $X \subseteq Y$ and $Y \cap Z = \varnothing$, then $X \cap Z = \varnothing$
For reductio, suppose $X \cap Z \neq \varnothing$, so that there is some $a \in X \cap Z$. So $a \in X$ and, since $X \subseteq Y$, we have that $a \in Y$. So $a \in Y \cap Z$, so that $Y \cap Z \neq \varnothing$. Contradiction! 4 marks