Philosophy Faculty Reading List and Course Outline 2019-2020

Part II Paper 07: Mathematical Logic

# Syllabus

* **First-order and second-order logic:** completeness, compactness, conservativeness, expressive power and Löwenheim-Skolem theorems.
* **First and second order theories:** categoricity, non-standard models of arithmetic.
* **Set theory:** embedding mathematics in set theory, the cumulative iterative hierarchy, elements of cardinal and ordinal arithmetic, the axiom of choice.
* **Recursive functions and computability:** decidability, axiomatizability, Church’s thesis, Gödel’s incompleteness theorems, Hilbert’s programme.

# Course Outline

This course aims to put the student in a position to assess the philosophical significance of some major results in mathematical logic. Alert attendance at lectures should be considered essential. But the emphasis is not only on the rigorous proof of results but also on philosophical reflection about them.

The first part of the course tackles the ideas of a formal logic and a formal theory with particular attention to the similarities and differences between first-order and second-order logic and arithmetic.

The next part studies set theory, its axioms and motivating conceptions, and the idea that all of mathematics can be embedded in set theory.

In the third part of the course on recursive functions the informal notion of a computable function is described and the relationship, embodied in Church's thesis, between this informal notion and the precise notion of a recursive function is examined. All this leads up to the understanding of Gödel's incompleteness theorems for arithmetic.

# Assumed Knowledge

The material in Part IB Paper 1 (Knowledge, Language and World) and Part IA Paper 5 (Formal Methods).

# Objectives

Students will be expected to:

* Study issues in mathematical logic at an advanced level.
* Acquire a sophisticated understanding of the scope, purpose and natures of logic.
* Refine their power of philosophical analysis and argument through study of these ideas.

# Preliminary Reading

## Basic logic

Chiswell, Ian, and Wilfrid Hodges, Mathematical Logic (Oxford: Oxford University Press, 2007). Also available online at: <https://ebookcentral.proquest.com/lib/cam/detail.action?docID=415527>.

Leary, Christopher C., A Friendly Introduction to Mathematical Logic (London: Prentice Hall, 2000); 2nd ed. (Geneseo, NY: Milne Library, SUNY Geneseo, 2015), chs. 1-3.

## Set theory

Button, Tim, 'Open Set Theory' (2019) [Online]. Available at: under the "OERs" section on <http://www.nottub.com> (Accessed: 14 August 2019).

Devlin, Keith, The Joy of Sets. 2nd ed. (New York: Springer-Verlag, 1993), chs. 1 & 2.

Goldrei, Derek, Classic Set Theory (London: Chapman & Hall/CRC, 1996).

Halmos, Paul R., Naïve Set Theory (New York: Springer, 1974).

## Arithmetic, computability etc.

Epstein, Richard L., and Walter Carnielli, Computability: Computable Functions, Logic, and the Foundations of Mathematics. 3rd ed. (Socorro, NY: Advanced Reasoning Forum, 2008). [You can skip the "optional" chapters]

For classic essays on some of the conceptual issues discussed in the course see:

Benacerraf, Paul, and Hilary Putnam, eds., Philosophy of Mathematics: Selected Readings. 2nd ed. (Cambridge: Cambridge University Press, 1983). Also available online at: <http://doi.org/10.1017/CBO9781139171519>.

# Reading List

## **Background**

### General formal surveys

Two very useful, more discursive surveys, standing back a bit from the nitty gritty of proofs, but trying to give a sense of how results fit together with an indication of their wider significance, are:

Rogers, Robert, Mathematical Logic and Formalized Theories (Amsterdam: North-Holland, 1971).

Wolf, Robert S., A Tour through Mathematical Logic (Washington: Mathematical Association of America, 2005).

Rogers’s now rather old book is very useful and very accessible though relatively introductory. Wolf's newer book goes further but is a rather bumpier ride because it's somewhat uneven in level of difficulty (though he gives some useful proof sketches).

These books will make very useful companions to formal work over the year, and could be especially helpful whenever you feel in danger of not seeing the wood for the trees.

### The Way the Rest of This Reading List is Structured

For formal expositions, we’ll list some possible alternatives, because different presentations are to the taste of different readers.

For philosophical topics, we usually distinguish core readings – in something like a sensible reading order – from a selection of possible further readings. Such divisions are inevitably somewhat arbitrary, and different supervisors will want to take different views about what is basic – needed to make a shot at a supervision essay – and what pushes on the debate rather further.

## First Order Logic

### Formal expositions

* You need to understand the following theorems:
* The Soundness and Completeness Theorem
* The Compactness Theorem
* The Löwenheim-Skolem Theorems

You also need to understand how to prove them.

The books by Chiswell/Hodges and by Leary already mentioned of course cover first-order logic in an accessible way. And almost any standard middle or advanced level text will cover the needed ground. But a stand-out presentation is:

Hodges, Wilfrid, 'Elementary Predicate Logic', in D. Gabbay and F. Guenthner, eds., Handbook of Philosophical Logic. Vol. 1 (Dordrecht: Reidel, 1984-89), pp. 1-131. Also an expanded version of this appears in the 2nd edition of the *Handbook.*

For a more discursive introduction to some main ideas see:

Rogers, Robert, Mathematical Logic and Formalized Theorie*s* (Amsterdam: North-Holland, 1971), chs. 2 & 3.

For another overview treatment highlighting the main ideas, though in more detail, you could see:

Boolos, George,John Burgess, and Richard C. Jeffrey, Computability and Logic. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007), chs. 9, 10 & 12-14. The 5th ed. is also available online at: <https://ebookcentral.proquest.com/lib/cam/detail.action?docID=321467>.

And here are two more standard textbook treatments, for enthusiasts:

Enderton, Herbert B., Mathematical Introduction to Logic. 2nd ed. (San Diego, CA: Harcourt/Academic Press, 2002), ch. 2 'First-order logic'.

Mendelson, Elliott, Introduction to Mathematical Logic. 4th ed. (Pacific Grove, CA: Wadsworth, 1997), sects. 2.1-2.9.

#### Philosophical issues arising: Skolem's paradox

For basic discussion see:

Bays, Timothy, 'Skolem's Paradox', in E.N. Zalta, ed., The Stanford Encyclopedia of Philosophy (Winter 2014 Edition) [Online]. Available at: <http://plato.stanford.edu/archives/win2014/entries/paradox-skolem/> (Accessed: 14 August 2019).

Giaquinto, Marcus, The Search for Certainty (Oxford: Clarendon Press, 2002), ch. 4, sect 2 'Blitz on paradise', pp. 130-36. Also available online at: <https://ebookcentral.proquest.com/lib/cam/reader.action?docID=4963696&ppg=142>.

For further reading, look at two absolutely classic papers (though they are **hard**):

Skolem, Thoralf, 'Some Remarks on Axiomatised Set Theory', in J. van Heijenoort, ed., From Frege to Gödel: A Source Book in Mathematical Logic, 1879-1931 (Cambridge, MA: Harvard University Press, 1967), pp. 290-301. **Also available on Moodle**. [Especially sect. 3]

Putnam, Hilary, 'Models and Reality', *Journal of Symbolic Logic*, 45, no. 3 (1980): 464-82. <https://doi.org/10.2307/2273415>. Also in his Realism and Reason, Philosophical Papers, Vol. 3 (Cambridge: Cambridge University Press, 1983), pp. 1-25. Also available online at:

<http://doi.org/10.1017/CBO9780511625275.003>. [Do not be alarmed if the “Epistemological/Logical Digression” makes no sense to you!]

Skolem’s paper is the *locus classicus* of the paradox; Putnam rejuvenated philosophical discussion of the paradox. To start understanding them, read the remainder of Bays’s ‘Skolem’s Paradox’, i.e. sections 3-4. Then look into:

Benacerraf, Paul, and Crispin Wright, 'Skolem and the Skeptic', Aristotelian Society, Suppl. Vol., 59 (1985): 117-37. <http://www.jstor.org/stable/4106752>. Reprinted in S. Shapiro, ed., *The Limits of Logic* (Aldershot: Dartmouth, 1996), pp.451-71.

Button, Tim, and Sean Walsh, Philosophy and Model Theory (Oxford: Oxford University Press, 2018), chs. 6-8, focussing on chapter 7. Also available online at: <https://doi.org/10.1093/oso/9780198790396.001.0001>.

George, Alexander, 'Skolem and the Lowenheim-Skolem Theorem: A Case Study of the Philosophical Significance of Mathematical Results', History and Philosophy of Logic, 6, no. 1 (1985): 75-89. <http://doi.org/10.1080/01445348508837077>. Also available online at: <http://doi.org/10.1080/01445348508837077>.

Klenk, Virginia, 'Intended Models and the Löwenheim-Skolem Theorem', Journal of Philosophical Logic, 5, no. 4 (1976): 475-89. <https://doi.org/10.1007/BF02109439>

Mcintosh, Clifton, 'Skolem's Criticisms of Set Theory', Noûs, 13, no. 3 (1979): 313-34. <https://doi.org/10.2307/2215103>

Melia, Joseph, 'The Significance of Non-Standard Models', Analysis, 55, no. 3 (1995): 127-34. <https://doi.org/10.2307/3328568>

Moore, Adrian W., The Infinite (London: Routledge, 1990), ch. 11 'The Löwenheim-Skolem theorem'. **Also available on Moodle**.

For further discussion, try:

Shapiro, Stewart, Foundations without Foundationalism (Oxford: Oxford University Press, 1991), ch. 8 'Second Order Logic and Rule Following'. Also available online at: <http://doi.org/10.1093/0198250290.003.0008>.

Tymoczko, Thomas, 'In Defense of Putnam’s Brains', Philosophical Studies, 57, no. 3 (1989): 281-97. <https://doi.org/10.1007/BF00372698>.

Wright, Crispin, 'Skolem and the Skeptic', Aristotelian Society, Suppl. Vol., 59 (1985): 85-137. <http://www.jstor.org/stable/4106752>.

## Second Order Logic

### Formal Expositions

You need some sense of the difference between first-order logic (full) and second-order logic in terms of axiomatizability, compactness, the Löwenheim-Skolem theorems, etc. You’ll need to understand why e.g. second-order Peano arithmetic is categorical (with the full semantics) and first-order Peano arithmetic isn't.

For a useful introductory overview, see:

Enderton, Herbert B., 'Second Order and Higher-Order Logic', in E.N. Zalta, ed., The Stanford Encyclopedia of Philosophy (Fall 2015 Edition) [Online]. Available at: <http://plato.stanford.edu/archives/fall2015/entries/logic-higher-order/> (Accessed: 14 August 2019).

The classic modern presentation is Shapiro’s, and this will give you more than enough:

Shapiro, Stewart, Foundations without Foundationalism (Oxford: Oxford University Press, 1991), chs. 3-5. Also available online at: <http://doi.org/10.1093/0198250290.001.0001>.

That said, you might find Shapiro’s approach to formal matters a little unfamiliar. So, for alternative treatments – including the contrast between first and second-order logic – dip into any of the following:

Boolos, George, John Burgess, and Richard Jeffrey, Computability and Logic. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007), ch. 22 'Second-order logic'. The 5th ed. is also available online at: <https://ebookcentral.proquest.com/lib/cam/reader.action?docID=321467&ppg=295>.

Button, Tim, and Sean Walsh, Philosophy and Model Theory (Oxford: Oxford University Press, 2018), ch. 1, and sects. 4.1, 7.3–7.6. Also available online at: <https://doi.org/10.1093/oso/9780198790396.001.0001>.

Van Dalen, Dirk, Logic and Structure. 5th ed. (London: Springer, 2013), ch. 5 'Second order logic'.

Finally, for more specific discussion of first- vs. second-order arithmetic, consider:

Smith, Peter, An Introduction to Gödel's Theorems (Cambridge: Cambridge University Press, 2007; 2nd ed., 2013), ch. 21 (ch. 24 in 2nd ed.) 'The Diagonalization Lemma'. 2nd ed. also available online at: <https://doi.org/10.1017/CBO9781139149105.025>.

### Philosophical issues arising: 1. On the status of second-order logic as logic

Is second-order logic just set-theory in disguise (with the second-order quantifiers running over sets)? That’s the view of:

Quine, Willard V.O., Philosophy of Logic. 2nd ed. (Cambridge, MA: Harvard University Press, 1986), ch. 5 'The scope of logic'. **Also available on Moodle.**

For discussion see:

Boolos, George, 'On Second-Order Logic', Journal of Philosophy, 72, no. 16 (1975): 502-26. <https://doi.org/10.2307/2025179>. Also in his *Logic, Logic and Logic* (Cambridge, MA: Harvard University Press, 1998); and in S. Shapiro, ed., The Limits of Logic (Aldershot: Dartmouth, 1996).

Shapiro, Stewart, Foundations without Foundationalism (Oxford: Oxford University Press, 1991), ch. 2, sects. 3-5. Also available online at: <https://doi.org/10.1093/0198250290.003.0002>.

For further reading, see:

Boolos, George, 'A Curious Inference', Journal of Philosophical Logic, 16, no. 1 (1987): 1-12. <http://www.jstor.org/stable/30226368>. Also in hisLogic, Logic and Logic(Cambridge, MA: Harvard University Press, 1998); and in S. Shapiro, ed., The Limits of Logic (Aldershot: Dartmouth, 1996).

Tharp, Leslie H., 'Which Logic Is the Right Logic?' Synthese, 31, no. 1 (1975): 1-21. <http://www.jstor.org/stable/20115054>. Also in S. Shapiro, ed., The Limits of Logic (Aldershot: Dartmouth, 1996).

Trueman, Robert, 'Neutralism within the Semantic Tradition', Thought, 1, no. 3 (2012): 246-51. <http://doi.org/10.1002/tht3.41>.

Väänänen, Jouko, 'Second-Order Logic and the Foundations of Mathematics', The Bulletin of Symbolic Logic, 7, no. 4 (2001): 504-20. <https://doi.org/10.2307/2687796>.

### Philosophical issues arising: 2. The connections with plural quantification and natural language

George Boolos has argued that we can “tame” second-order logic (and see it as genuinely part of logic) by interpreting second-order quantifiers as (akin to) plural quantifiers. For a basic exchange, see:

Boolos, George, 'To Be Is to Be a Value of a Variable (or to Be Some Values of Some Variables)', Journal of Philosophy, 81, no. 8 (1984): 430-49. <https://doi.org/10.2307/2026308>. Reprinted in his Logic, Logic and Logic(Cambridge, MA: Harvard University Press, 1998); and in S. Shapiro, ed., The Limits of Logic (Aldershot: Dartmouth, 1996).

Resnik, Michael D., 'Second Order Logic Still Wild', Journal of Philosophy, 85, no. 2 (1988): 75-87. <https://doi.org/10.2307/2026993>. Reprinted in S. Shapiro, ed., The Limits of Logic (Aldershot: Dartmouth, 1996).

For further discussion see:

Boolos, George, 'Nominalist Platonism', Philosophical Review, 94, no. 3 (1985): 327-44. <https://doi.org/10.2307/2185003>. Reprinted in hisLogic, Logic and Logic(Cambridge, MA: Harvard University Press, 1998); and in S. Shapiro, ed., The Limits of Logic (Aldershot: Dartmouth, 1996).

Higginbotham, James, 'On Higher-Order Logic and Natural Language', in T.J. Smiley, ed., Philosophical Logic (Oxford: Oxford University Press, 1998), pp. 1-27.

Linnebo, Øystein, 'Plural Quantification', in E.N. Zalta, ed., Stanford Encyclopedia of Philosophy (Summer 2017 edition) [Online]. Available at: <http://plato.stanford.edu/archives/sum2017/entries/plural-quant/> (Accessed: 14 August 2019).

Rayo, Agustin, and Stephen Yablo, 'Nominalism through De-Nominalization', Noûs, 35, no. 1 (2001): 74-92. <http://www.jstor.org/stable/2671946>.

## Set theory

Start with this book, which covers all the technicalities:

Button, Tim, 'Open Set Theory' (2019) [Online]. Available at: under the "OERs" section on <http://www.nottub.com> (Accessed: 14 August 2019).

It will also give you a quick introduction to several of the philosophical issues.

### Formal expositions

Different people will respond to different formal texts; so if you want alternative treatments, these texts outline some key ideas behind ZFC:

George, Alexander, and Daniel J. Velleman, Philosophies of Mathematics (Oxford: Blackwell, 2002), ch. 3 ‘Set theory’.

Wolf, Robert S., A Tour through Mathematical Logic (Washington, DC: Mathematical Association of America, 2005), ch. 2 'Axiomatic set theory'.

The next three textbooks are excellent, fuller treatments of axiomatic set theory:

Devlin, Keith, The Joy of Sets. 2nd ed. (New York: Spinger, 1993), chs. 1-3.

Goldrei, Derek, Classic Set Theory (London: Chapman & Hall, 1996).

Halmos, Paul R., Naïve Set Theory (New York: Springer, 1974).

Here's another modern text that is written in a relaxed style, and is often extremely helpful in the way it introduces concepts and theorems:

Just, Winfried, and Martin Weese, *Discovering Modern Set Theory I* (Providence, RI: American Mathematical Society, 1996).

Finally, a lovely way of thinking about set theory – coming to be known as "the Scott-Potter theory" – is presented in the following text (though be warned that this is indeed a somewhat unusual approach to the technicalities):

Potter, Michael, Set Theory and Its Philosophy (Oxford: Oxford University Press, 2003). Also available online at: <https://ebookcentral.proquest.com/lib/cam/detail.action?docID=422705>.

### Historical background

If you want to know about the history of ZFC, try looking at:

Ferreirós, José, 'The Early Development of Set Theory', in E.N. Zalta, ed., The Stanford Encyclopedia of Philosophy (Summer 2019 Edition) [Online]. Available at: <http://plato.stanford.edu/archives/sum2019/entries/settheory-early/> (Accessed: 14 August 2019).

Fraenkel, Abraham A., and Yehoshua Bar-Hillel, Foundations of Set Theory (Amsterdam: North-Holland, 1958), chs. 1 & 2.

Zermelo, Ernst, 'Investigations in the Foundations of Set Theory I', in J. van Heijenoort, ed., From Frege to Gödel: A Source Book in Mathematical Logic*, 1879-1931* (Cambridge, MA: Harvard University Press, 1967), pp. 199-215.

Those who find the historical stories fascinating - and they illuminate why one particular set theory has ended up as the canonical one - can follow up Ferreirós by dipping into at least the first half of the longer story as told by:

Kanamori, Akihiro, 'The Mathematical Development of Set Theory from Cantor to Cohen', Bulletin of Symbolic Logic, 2, no. 1 (1996): 1-71. <https://doi.org/10.2307/421046>

### Philosophical issues arising: 1. Set theory as a foundation for mathematics

In what sense can we say that set theory "provides a foundation for" mathematics?

Giaquinto, Marcus, The Search for Certainty (Oxford: Oxford University Press, 2002), part I; and part V, sects. 1 & 2. Also available online at: <https://ebookcentral.proquest.com/lib/cam/detail.action?docID=4963696>.

Gives an introductory discussion of what set theory is supposed to do for us. The set theory texts above have things to say as they go along. For further discussion see:

Maddy, Penelope, Naturalism in Mathematics (Oxford: Oxford University Press, 1997), Part 2 'Realism'. Also available online at: <http://doi.org/10.1093/0198250754.001.0001>.

Mayberry, John, 'What Is Required of a Foundation for Mathematics?' Philosophia Mathematica, 2, no. 1 (1994): 16-35. <http://doi.org/10.1093/philmat/2.1.16>. Reprinted in D. Jacquette, ed., Philosophy of Mathematics: An Anthology (Oxford: Blackwell, 2002).

There are some subversive remarks too in:

Oliver, Alex, and Timothy Smiley, 'What Are Sets and What Are They For?' *Philosophical Perspectives*, 20, Metaphysics (2006): 123-55. <http://doi.org/10.1111/j.1520-8583.2006.00105.x>

### Philosophical issues arising: 2. What conception of sets is supposed to be reflected in standard set theories? Does that conception justify the axioms?

Boolos, George, 'The Iterative Conception of Set', Journal of Philosophy, 68, no. 8 (1971): 215-31. <https://doi.org/10.2307/2025204>. Also in his Logic, Logic and Logic (Cambridge, MA: Harvard University Press, 1998).

Boolos, George, 'Iteration Again', Philosophical Topics, 17 (1989): 5-21. <http://doi.org/10.5840/philtopics19891721>. Also in his Logic, Logic and Logic (Cambridge, MA: Harvard University Press, 1998).

Parsons, Charles, 'What Is the Iterative Conception of Set?' in P. Benacerraf and H. Putnam, eds., Mathematics in Philosophy: Selected Essays. 2nd ed. (Cambridge: Cambridge University Press, 1983), pp. 503-29. Also available online at: <https://doi.org/10.1017/CBO9781139171519.027>.

See also:

Forster, Thomas, 'The Iterative Conception of Set', Review of Symbolic Logic, 1, no. 1 (2008): 97-110. <http://doi.org/10.1017/S1755020308080064>.

Gödel, Kurt, 'What Is Cantor's Continuum Problem?' American Mathematical Monthly, 54, no. 9 (1947): 515-25. <https://doi.org/10.2307/2304666>. Also in his *Collected Works*, Vol. II (Oxford: Oxford University Press, 1990); and in P. Benacerraf & H. Putnam, eds., Philosophy of Mathematics: Selected Readings. 2nd ed. (Cambridge: Cambridge University Press, 1983). Also available online at: <https://doi.org/10.1017/CBO9781139171519.025>.

Paseau, Alexander, 'Boolos on the Justification of Set Theory', Philosophia Mathematica, 15, no. 1 (2007): 30-53. <http://doi.org/10.1093/philmat/nkl017>

Potter, Michael, 'Iterative Set Theory', Philosophical Quarterly, 43, no. 171 (1993): 178-93. <https://doi.org/10.2307/2220368>

Wang, Hao, From Mathematics to Philosophy (London: Routledge & Kegan Paul, 1974), 'The concept of set'. Reprinted in P. Benacerraf & H. Putnam, eds., Philosophy of Mathematics: Selected Readings, 2nd ed. (Cambridge: Cambridge University Press, 1983), pp 530-70. Also available online at: <https://doi.org/10.1017/CBO9781139171519.028>.

## Gödel's First Incompleteness Theorem

### Formal expositions

For a nice introduction (in a splendidly sane short book, which you should eventually read all of), see:

Franzén, Torkel, Gödel's Theorem: An Incomplete Guide to Its Use and Abuse (Wellesley: A.K. Peters, 2005), chs. 1-3.

And for another introductory survey see:

Rogers, Robert, Mathematical Logic and Formalized Theories (Amsterdam: North-Holland, 1971), ch. 8 'Incompleteness. Undecidability'.

There's a bit more detail again in:

George, Alexander, and Daniel J. Velleman, Philosophies of Mathematics (Oxford: Blackwell, 2001), ch. 7 'The Incompleteness Theorems'.

But for a full-dress proof with all the trimmings see:

Smith, Peter, An Introduction to Gödel's Theorems (Cambridge: Cambridge University Press, 2007; 2nd ed., 2013). 2nd ed. also available online at: <http://doi.org/10.1017/CBO9781139149105>. [Especially chs .16 &17 (chs. 21 & 22 in 2nd ed.); those chapters more or less follow Gödel's original proof]

**Note:** Gödel proved his First Theorem in 1931, before the beginnings of the general theory of computability really got underway in 1936: the original version of the Theorem appeals only to the restricted notion of a "primitive recursive" function. Many modern books, however, approach things in a non-historical order, first explaining the general theory of computability, and then moving on to Gödel's Theorem. Two notable books which do things this way around are:

Boolos, George,John Burgess, and Richard C. Jeffrey, Computability and Logic. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007). The 5th ed. is also available online at: <https://ebookcentral.proquest.com/lib/cam/reader.action?docID=321467&ppg=236>. [Gets to Gödel's Theorem in ch. 17]

Epstein, Richard L., and Walter Carnielli, Computability: Computable Functions, Logic and the Foundations of Mathematics. 3rd ed. (Socorro, NY: Advanced Reasoning Forum, 2008).

### Philosophical issues arising: 1. Minds and machines

Lucas, John R., 'Minds, Machines and Gödel', Philosophy, 36, no. 137 (1961): 112-27. <http://www.jstor.org/stable/3749270>. Reprinted in A.R. Anderson, ed., Minds and Machines (Englewood Cliffs, NJ: Prentice-Hall, 1964), pp. 43-59.

Lucas famously argues that Gödel's theorem shows that minds are not machines. (It is not really essential, but might help if you know what Turing machine is before you start reading this debate). For a classic riposte, see:

Putnam, Hilary, Minds and Machines: Mind, Language and Reality, Philosophical Papers Vol. 2 (Cambridge: Cambridge University Press, 1975). Also available online at: <https://doi.org/10.1017/CBO9780511625251.020>.

Others have tried to rescue Lucas's argument, in particular:

Penrose, Roger, Shadows of the Mind (Oxford: Oxford University Press, 1994), chs. 2 & 3, esp.sects. 2.5-3.10.

For a stern critique of that see:

Feferman, Soloman, 'Penrose's Gödelian Argument', Psyche, 2, no. 7 (1995): 21-32. Also available online at: <http://journalpsyche.org/files/0xaa23.pdf>.

(There's much more on Penrose to be found in the same issue of Psyche at: <http://journalpsyche.org/archive/volume-2-1995-1996/>)

For other related discussion see:

Gödel, Kurt, Collected Works. Vol. III (Oxford: Oxford University Press, 1995), pp. 304-23 'Some basic theorems in the foundations of mathematics and their philosophical implications'. **Also available on Moodle**. [This, the "Gibbs Lecture" from 1951 is not easy but is remarkably rich]

Lewis, David, 'Lucas against Mechanism', Philosophy, 44, no. 169 (1969): 231-33. <http://www.jstor.org/stable/3749666>. Reprinted in his *Papers in* Philosophical Logic (Cambridge: Cambridge University Press, 1998). Also available online at: <http://doi.org/10.1017/CBO9780511625237>.

Penrose, Roger, The Emperor's New Mind (Oxford: Oxford University Press, 1989). [Especially pp. 129-46 & 538-41 - this is Penrose's first shot at extracting philosophical morals from Gödel, in an earlier book]

Smith, Peter, An Introduction to Gödel's Theorems (Cambridge: Cambridge University Press, 2007; 2nd ed., 2013). 2nd ed. also available online at: <https://doi.org/10.1017/CBO9781139149105.038>. [Sect. 28.6 (37.6 in 2nd ed.) - relates to the argument in Gödel's paper which springs from the Second Incompleteness Theorem]

### Philosophical issues arising: 2. Is the notion of natural number open-ended?

Dummett, Michael, 'The Philosophical Significance of Gödel's Theorem', Ratio, 5 (1963): 140-55. Reprinted in his Truth and other Enigmas (London: Duckworth, 1978), pp. 186-201.

Moore, Adrian W., 'More on "The Philosophical Significance of Gödel's Theorem"', in J.L. Brandl and P.M. Sullivan, eds., New Essays on the Philosophy of Michael Dummett (Series: Grazer Philosophische Studien; 50) (Amsterdam: Rodopi, 1998), pp. 103-26. **Also available on Moodle.**

Wright, Crispin, Realism, Meaning and Truth. 2nd ed. (Oxford: Blackwell, 1993), ch. 11 'About "The philosophical significance of Gödel's theorem": Some issues'.

## Gödel's Second Incompleteness Theorem

### Formal expositions

George, Alexander, and Daniel J. Velleman, Philosophies of Mathematics (Oxford: Blackwell, 2001), ch. 7 'The Incompleteness Theorems'.

could be a useful place to start. And for quite a bit more detail see:

Smith, Peter, An Introduction to Gödel's Theorems (Cambridge: Cambridge University Press, 2007; 2nd, 2013), chs. 24-26 (chs. 31-35 in 2nd ed.). 2nd ed. also available online at: <http://doi.org/10.1017/CBO9781139149105>.

That should put you in a position to appreciate Boolos's wonderful *jeu d'esprit*:

Boolos, George, 'Gödel's Second Incompleteness Theorem Explained in Words of One Syllable', Mind, 103, no. 409 (1994): 1-3. <http://www.jstor.org/stable/2253954>. Also in his Logic, Logic and Logic (Cambridge, MA: Harvard University Press, 1998).

For useful commentary, see:

Moore, Adrian W., 'What Does Gödel's Second Incompleteness Theorem Show?' Noûs, 22, no. 4 (1988): 573-84. <https://doi.org/10.2307/2215458>

## Hilbert's Programme

As we'll see, the main philosophical issue arising from Gödel's Second Theorem (at least as far as this paper is concerned) is its impact on Hilbert's Programme. For a very good introduction to Hilbert, see:

Giaquinto, Marcus, The Search for Certainty (Oxford: Oxford University Press, 2002), part IV, sects. 3 & 4. Also available online at: <https://ebookcentral.proquest.com/lib/cam/reader.action?docID=4963696&ppg=129>.

But do read the man himself:

Hilbert, David, 'On the Infinite', in P. Benacerraf and H. Putnam, eds., Philosophy of Mathematics: Selected Readings (Oxford: Blackwell, 1964; 2nd ed. Cambridge University Press, 1983), pp. 183-201. Also available online at: <https://doi.org/10.1017/CBO9781139171519.010>. Reprinted in J. van Heijenoort, ed., From Frege to Gödel: a Source Book in Mathematical Logic, 1879-1931 (Cambridge, MA: Harvard University Press, 1967).

For further elaboration see also:

George, Alexander, and Daniel J. Velleman, Philosophies of Mathematics (Oxford: Blackwell, 2001), ch. 6 'Finitism'.

And for further discussion, see:

Kreisel, Georg, 'Hilbert's Programme', Dialectica, 12, no. 3-4 (1958): 346-72. <http://doi.org/10.1111/j.1746-8361.1958.tb01469.x>. Reprinted in P. Benacerraf and H. Putnam (eds.), Philosophy of Mathematics: Selected Readings. 2nd ed. (Cambridge: Cambridge University Press, 1983). Also available online at: <https://doi.org/10.1017/CBO9781139171519.012>.

Parsons, Charles, 'Finitism and Intuitive Knowledge', in M. Schirn, ed., The Philosophy of Mathematics Today (Oxford: Oxford University Press, 1998), pp. 249-70. **Also available on Moodle.**

Potter, Michael D., Reason's Nearest Kin (Oxford: Oxford University Press, 2000), ch. 9 'Hilbert's Programme'. Also available online at: <http://doi.org/10.1093/acprof:oso/9780199252619.003.0010>.

Tait, William W., 'Finitism', Journal of Philosophy, 78, no. 9 (1981): 524-46. <https://doi.org/10.2307/2026089>

### Philosophical issue arising: What was Hilbert's Programme? Do Gödel's incompleteness theorems undermine it?

A standard answer to the second question is given by:

Giaquinto, Marcus, The Search for Certainty (Oxford: Oxford University Press, 2002), ch. 5, sect. 2 'Underivability of 'Consistency''. Also available online at: <https://ebookcentral.proquest.com/lib/cam/reader.action?docID=4963696&ppg=194>.

And similarly by:

Smith, Peter, An Introduction to Gödel's Theorems (Cambridge: Cambridge University Press, 2007; 2nd ed, 2013), sects. 28.1-28.5 (sects. 37.1-37.5 in 2nd ed.). 2nd ed. also available online at: <https://doi.org/10.1017/CBO9781139149105.038>.

For dissent see:

Detlefsen, Michael, 'On an Alleged Refutation of Hilbert's Program Using Gödel's First Incompleteness Theorem', Journal of Philosophical Logic, 19, no. 4 (1990): 343-77. <http://www.jstor.org/stable/30226438>. Reprinted in his Proof, Logic and Formalization (London: Routledge, 1991), ch. 8.

For more discussions see:

Gentzen, Gerhard, The Collected Papers of Gerhard Gentzen, edited by M.E. Szabo (Amsterdam: North-Holland, 1969), ch.6 'The concept of infinity in mathematics'.

Raatikainen, Panu, 'Hilbert's Program Revisited', Synthese, 137, no. 1-2 (2003): 157-77. <http://www.jstor.org/stable/20118356>.

Simpson, Stephen G., 'Partial Realizations of Hilbert's Program', Journal of Symbolic Logic, 53, no. 2 (1988): 349-63. <https://doi.org/10.2307/2274508>. [For enthusiasts who want to know something of the afterlife of Hilbert's Programme]

Zach, Richard, 'Hilbert's Program', in E.N. Zalta, ed., The Stanford Encyclopedia of Philosophy (Summer 2019 Edition) [Online]. Available at: <http://plato.stanford.edu/archives/sum2019/entries/hilbert-program/> (Accessed: 14 August 2019).

Zach, Richard, 'Hilbert's Program, Then and Now', in D. Jacquette, ed., Philosophy of Logic (Amsterdam: Elsevier, 2006), pp. 411-47. Also available online at: <http://philsci-archive.pitt.edu/2547/1/hptn.pdf>.

## Recursive Functions and Computability

### Expositions

What is Turing computable function? What is recursive function? Why are they the same class of functions? For explanations, see:

Smith, Peter, An Introduction to Gödel's Theorems (Cambridge: Cambridge University Press, 2007; 2nd ed.2013), chs. 29, 31 & 32 (chs. 38, 41 & 42 in 2nd ed.). 2nd ed. also available online at: <http://doi.org/10.1017/CBO9781139149105>.

For other alternatives, see:

Boolos, George, John Burgess, and Richard C. Jeffrey, Computability and Logic. 4th or 5th ed. (Cambridge: Cambridge University Press, 2002; 2007), chs. 3-8. The 5th ed. is also available online at: <https://ebookcentral.proquest.com/lib/cam/detail.action?docID=321467>. Though many think the treatment of the same chapters of the 3rd ed. - when the authors were just Boolos and Jeffrey - is nicer.

Cutland, Nigel, Computability: An Introduction to Recursive Function Theory (Cambridge: Cambridge University Press, 1980), chs. 1-5. [A classic book that will appeal to mathematicians]

Hamilton, Alan G., Logic for Mathematicians (Cambridge: Cambridge University Press, 1978), ch. 7 'Computability, unsolvability, undecidability'.

Rogers, Hartley, Theory of Recursive Functions and Effective Computability (Cambridge, MA: MIT Press, 1987). [Another old classic from 1967 which is well worth reading the first chapter of, especially sects. 1.1-1.7]

### Philosophical issue arising: What is the status of Church's thesis?

It is a mathematical theorem that a function is Turing computable if and only if it is recursive (if and only if it register computable, if and only if it is Herbrand-Gödel computable etc.). Different attempts to regiment the intuitive notion of a computable function all converge. Church's Thesis (a.k.a. the Church-Turing Thesis) claims that indeed the *intuitively* computable functions are just the Turing computable/recursive functions. For some initial clarifications, see:

Smith, Peter, An Introduction to Gödel's Theorems (Cambridge: Cambridge University Press, 2007; 2nd, 2013), ch. 32 (ch. 42 in 2nd ed.) 'Turing machines and recursiveness'. 2nd ed. also available online at: <https://doi.org/10.1017/CBO9781139149105.042>.

Then read:

Shapiro, Stewart, 'Understanding Church's Thesis', Journal of Philosophical Logic10, no. 3 (1981): 353-66. <http://www.jstor.org/stable/30226230>

Black, Robert, 'Proving Church's Thesis', Philosophia Mathematica, 8, no. 3 (2000): 244-58. <http://doi.org/10.1093/philmat/8.3.244>

Shapiro, Stewart, 'Computability, Proof and Open-Texture', in A. Olszewski, J. Wolenski and R. Janusz, eds., Church's Thesis after 70 Years (Heusenstaam: Ontos, 2006). Also available online at: <https://doi.org/10.1515/9783110325461.420>.

Smith, Peter, An Introduction to Gödel's Theorems (Cambridge: Cambridge University Press, 2007; 2nd ed., 2013). 2nd ed. also available online at: <https://doi.org/10.1017/CBO9781139149105.044>. [Ch. 33 (ch. 43 in 2nd ed.) 'Halting problems' (‘Halting and incompleteness’ in 2nd ed.), takes an opinionated minority line]

See also:

Copeland, B. Jack, 'The Church-Turing Thesis', in E.N. Zalta, ed., The Stanford Encyclopedia of Philosophy (Spring 2019 Edition) [Online]. Available at: <http://plato.stanford.edu/archives/spr2019/entries/church-turing/> (Accessed: 14 August 2019).

Mendelson, Elliott, 'Second Thoughts About Church's Thesis and Mathematical Proofs', Journal of Philosophy, 87, no. 5 (1990): 225-33. <https://doi.org/10.2307/2026831>

Smith, Peter, 'Review of *Church's Thesis after 70 Years* ' (2007) [Online]. Available at: <https://www.logicmatters.net/resources/pdfs/CTT.pdf> (Accessed: 14 August 2019).

There's an interesting local sub-debate here:

Hogarth, Mark, 'Non-Turing Computers and Non-Turing Computability', Proceedings of the Biennial Meetings of the Philosophy of Science Association, 1 (1994): 126-38. <http://www.jstor.org/stable/193018>

Hogarth, Mark, 'Deciding Arithmetic Using Sad Computers', British Journal for the Philosophy of Science, 55, no. 4 (2004): 681-91. <http://www.jstor.org/stable/3541622>

Button, Tim, 'Sad Computers and Two Versions of the Church-Turing Thesis', British Journal for the Philosophy of Science, 60, no. 4 (2009): 765-92. <https://doi.org/10.1093/bjps/axp038>. [Criticizes Hogarth]

**We welcome your suggestions for further readings that will improve and diversify our reading lists, to reflect the best recent research, and important work by members of under-represented groups. Please email your suggestions to** [**phillib@hermes.cam.ac.uk**](mailto:phillib@hermes.cam.ac.uk) **including the relevant part and paper number. For information on how we handle your personal data when you submit a suggestion please see:** [**https://www.information-compliance.admin.cam.ac.uk/data-protection/general-data**](https://www.information-compliance.admin.cam.ac.uk/data-protection/general-data)**.**