SETS RELATIONS & PROBABILITY LECTURE 1

1. What is a set? You can think of it as a big shopping basket containing things. You can put anything you like into a shopping basket. So while there are sets of things belonging to natural kinds—e.g. the set of all dogs—there are also sets composed of miscellaneous things e.g. the set of Trafalgar Square, Waterloo Station and the island of St Helena. If an object x belongs to a set Y then it is said to be a *member* of that set, written $x \in Y$.

2. If a and b are the only members of a set then we can write the set as {a, b}. The order is immaterial: we could write {b, a} if we liked. And the same goes for any (finite) list. Note that since anything can belong to a set, sets can belong to sets too. For instance there is a set {{a, b}, c}.

3. A set is *not* a mere fusion of the things in it. You don't have to stick together Trafalgar Square etc. to make the set mentioned above. Nor is it the mere sum of those physically disconnected objects in the way that the territory of the United States might be regarded as the sum of its mainland, Alaska and Hawaii.

4. That follows from the most basic law concerning sets, the *axiom of extensionality*: sets are identical if and only if they have the same members. In symbols this is written:

   $$\forall X \forall Y ((SX \wedge SY) \rightarrow (X = Y \leftrightarrow \forall z (z \in X \leftrightarrow z \in Y)))$$

   Here 'S' is the predicate 'is a set'. It follows from the axiom of extensionality that a set is not the fusion or sum of its members. For if it were then the set of all states in the United States would be the same as the set of all counties in the United States. But the axiom of extensionality tells us that this is false: Florida belongs to the former but not the latter.

5. A second law concerning sets is that if X and Y are sets then there is a set containing the things that are in both X and Y. This is known as $X \cap Y$ or the *intersection* of X and Y. In symbols:

   $$\forall X \forall Y \forall z (z \in X \cap Y \leftrightarrow (z \in X \wedge z \in Y))$$

   For example, if X is the set of all cats and Y is the set of all pets then $X \cap Y$ is the set of all pet cats. Also: if X and Y are sets then there is a set containing all the things that are either X or in Y. This is called the *union* of X and Y and written $X \cup Y$. In symbols:

   $$\forall X \forall Y \forall z (z \in X \cup Y \leftrightarrow (z \in X \vee z \in Y))$$

   For example, if X is the set of all children in the class with blue eyes and Y is the set of all children in the class without blue eyes then their union is the set of all children in the class. Think of it as taking two shopping baskets and

pouring the contents of both into one big shopping basket. We use Venn diagrams to represent unions and intersection.

6. The other relation between sets that you need to know about is called inclusion or subsethood. Inclusion is NOT the same thing as membership. We say that X is included in Y, or X is a subset of Y, written X ⊆ Y, just in case all the members of X are members of Y. In symbols:

$$\forall X \forall Y ((SX \wedge SY) \rightarrow (X \subseteq Y \leftrightarrow \forall z (z \in X \rightarrow z \in Y)))$$

For example, the set of all donkeys is a subset of the set of all animals because all donkeys are animals. In fact the set of all animals is a subset of the set of all animals because all animals are animals. This too can be represented using Venn diagrams. We may also distinguish between inclusion and *proper* inclusion. We say that X is a proper subset of Y, which is written X ⊂ Y, just in case every member of X is a member of Y but there are some things in Y that are not in X i.e. X is a subset of Y but Y is not a subset of X. For example, the set of all donkeys is in fact a proper subset of the set of all animals for there are animals that are not donkeys, though all donkeys are animals.

7. Today I'll look at one more putative law that turns the subject from a philosophical triviality into a complete mystery: the comprehension principle. This says that for any condition you like, there is a set consisting of just the objects that satisfy this condition. That is, for any condition F there is a set S such that: $\forall x (x \in S \leftrightarrow Fx)$.

8. So there is a set of all things x satisfying the condition 'x is a King of France' and one consisting of all those things satisfying the condition 'x is a set with more than one member'; there is also a set of everything i.e. that satisfying the condition x = x. Notice from the second case that sets can be members of sets too. The troubling thing about this axiom is that it is inconsistent: the demonstration of this is known as Russell's Paradox.

9. The first two examples in no. 8 illustrate that some sets are, and some sets are not, members of themselves. Similarly, the set of all the examples that I used in this lecture is a member of itself because it is itself an example that I have used in this lecture. But the set of all donkeys is not a member of itself because the set of all donkeys is not a donkey. The comprehension principle says that there is a set *of all sets that are not members of themselves*. These are just the sets satisfying the condition $\neg(x \in x)$. But is *this* set a member of itself? Well if it is, then it must satisfy the condition $\neg(x \in x)$, which says that it is *not* a member of itself. But if it *isn't* a member of itself then it must satisfy $\neg(x \in x)$ and therefore *is* a member of itself. So it is a member of itself if and only if it is not. What is truly mysterious is that inconsistency can result from such an intuitive and seemingly obvious way of arranging things.