1A Logic — 2015 Model Answers

Section A

1. (a) Show each of the following [40]:

(i) $P \lor Q, \neg Q \lor R, \neg P \to \neg R \vdash P$

1	$P \lor Q$	
2	$\neg Q \lor R$	
3	$\neg P \to \neg R$	
4	$\quad \mid P$	
5	$\quad \mid P$	R, 4
6	$\quad \mid \neg P$	
7	$\quad \mid Q$	DS, 1, 6
8	$\quad \mid R$	DS, 2, 7
9	$\quad \mid \quad \mid \neg R$	
10	$\quad \mid \quad \mid \bot$	$\bot$I, 8, 9
11	$\quad \mid \neg\neg R$	$\neg$E, 9, 10
12	$\quad \mid \neg\neg P$	MT, 3, 11
13	$\quad \mid P$	DNE, 12
14	P	TND, 4–5, 6–13

(ii) $A \leftrightarrow \neg B \vdash (B \wedge \neg A) \vee (A \wedge \neg B)$

1	$A \leftrightarrow \neg B$	
2	$\quad A$	
3	$\quad \neg B$	$\leftrightarrow$E, 1, 2
4	$\quad A \wedge \neg B$	$\wedge$I, 2, 3
5	$\quad (B \wedge \neg A) \vee (A \wedge \neg B)$	$\vee$I, 4
6	$\quad \neg A$	
7	$\quad\quad \neg B$	
8	$\quad\quad A$	$\leftrightarrow$E, 1, 7
9	$\quad\quad \bot$	$\bot$I, 6, 8
10	$\quad \neg\neg B$	$\neg$I, 7–9
11	$\quad B$	DNE, 10
12	$\quad B \wedge \neg A$	$\wedge$I, 6, 11
13	$\quad (B \wedge \neg A) \vee (A \wedge \neg B)$	$\vee$I, 12
14	$(B \wedge \neg A) \vee (A \wedge \neg B)$	TND, 2–5, 6–13

(iii) $(A \wedge B) \to \neg C, B \leftrightarrow C, (C \wedge A) \vee (\neg C \wedge \neg A) \vdash \neg A$

1	$(C \wedge A) \vee (\neg C \wedge \neg A)$	
2	$B \leftrightarrow C$	
3	$(A \wedge B) \to \neg C$	
4	$\mid\ C \wedge A$	
5	$\mid\ C$	$\wedge$E, 4
6	$\mid\ B$	$\leftrightarrow$E, 2, 5
7	$\mid\ A$	$\wedge$E, 4
8	$\mid\ A \wedge B$	$\wedge$I, 6, 7
9	$\mid\ \neg C$	$\to$E, 3, 8
10	$\mid\ \bot$	$\bot$I, 5, 9
11	$\neg(C \wedge A)$	$\neg$I, 4–10
12	$(\neg C \wedge \neg A)$	DS, 1, 11
13	$\neg A$	$\wedge$E, 12

(iv) $\vdash (P \leftrightarrow Q) \to (\neg P \leftrightarrow \neg Q)$

1	$\mid\ P \leftrightarrow Q$	
2	$\mid\ \mid\ \neg P$	
3	$\mid\ \mid\ \mid\ Q$	
4	$\mid\ \mid\ \mid\ P$	$\leftrightarrow$E, 1, 3
5	$\mid\ \mid\ \mid\ \bot$	$\bot$I, 2, 4
6	$\mid\ \mid\ \neg Q$	$\neg$I, 3–5
7	$\mid\ \mid\ \neg Q$	
8	$\mid\ \mid\ \mid\ P$	
9	$\mid\ \mid\ \mid\ Q$	$\leftrightarrow$E, 1, 8
10	$\mid\ \mid\ \mid\ \bot$	$\bot$I, 7, 9
11	$\mid\ \mid\ \neg P$	$\neg$I, 8–10
12	$\mid\ \neg P \leftrightarrow \neg Q$	$\leftrightarrow$I, 2–6, 7–11
13	$(P \leftrightarrow Q) \to (\neg P \leftrightarrow \neg Q)$	$\to$I, 1–12

(v) $\vdash (P \vee Q) \rightarrow (((P \wedge Q) \vee (P \wedge \neg Q)) \vee (Q \wedge \neg P))$

1	$\mid\ \mid\ P \vee Q$	
2	$\mid\ \mid\ \mid\ P$	
3	$\mid\ \mid\ \mid\ \mid\ Q$	
4	$\mid\ \mid\ \mid\ \mid\ P \wedge Q$	$\wedge$I, 2–3
5	$\mid\ \mid\ \mid\ \mid\ (P \wedge Q) \vee (P \wedge \neg Q)$	$\vee$I, 4
6	$\mid\ \mid\ \mid\ \mid\ (((P \wedge Q) \vee (P \wedge \neg Q)) \vee (Q \wedge \neg P))$	$\vee$I, 5
7	$\mid\ \mid\ \mid\ \mid\ \neg Q$	
8	$\mid\ \mid\ \mid\ \mid\ P \wedge \neg Q$	$\wedge$I, 2, 7
9	$\mid\ \mid\ \mid\ \mid\ (P \wedge Q) \vee (P \wedge \neg Q)$	$\vee$I, 8
10	$\mid\ \mid\ \mid\ \mid\ (((P \wedge Q) \vee (P \wedge \neg Q)) \vee (Q \wedge \neg P))$	$\vee$I, 9
11	$\mid\ \mid\ \mid\ (((P \wedge Q) \vee (P \wedge \neg Q)) \vee (Q \wedge \neg P))$	TND, 3–6, 7–10
12	$\mid\ \mid\ \mid\ Q$	
13	$\mid\ \mid\ \mid\ \mid\ P$	
14	$\mid\ \mid\ \mid\ \mid\ P \wedge Q$	$\wedge$I, 12–13
15	$\mid\ \mid\ \mid\ \mid\ (P \wedge Q) \vee (P \wedge \neg Q)$	$\vee$I, 14
16	$\mid\ \mid\ \mid\ \mid\ (((P \wedge Q) \vee (P \wedge \neg Q)) \vee (Q \wedge \neg P))$	$\vee$I, 15
17	$\mid\ \mid\ \mid\ \mid\ \neg P$	
18	$\mid\ \mid\ \mid\ \mid\ Q \wedge \neg P$	$\wedge$I, 12, 17
19	$\mid\ \mid\ \mid\ \mid\ (((P \wedge Q) \vee (P \wedge \neg Q)) \vee (Q \wedge \neg P))$	$\vee$I, 18
20	$\mid\ \mid\ \mid\ (((P \wedge Q) \vee (P \wedge \neg Q)) \vee (Q \wedge \neg P))$	TND, 13–16, 17–19
21	$\mid\ \mid\ (((P \wedge Q) \vee (P \wedge \neg Q)) \vee (Q \wedge \neg P))$	$\vee$E, 2–11, 12–20
22	$\mid\ (P \vee Q) \rightarrow (((P \wedge Q) \vee (P \wedge \neg Q)) \vee (Q \wedge \neg P))$	$\rightarrow$I, 1–21

(b) Show each of the following [60]:

(i) $\exists y(Ryy \leftrightarrow Ga) \vdash (\neg\forall x \neg Rxx) \lor \neg Ga$

1	$\exists y(Ryy \leftrightarrow Ga)$	
2	$\quad Rbb \leftrightarrow Ga$	
3	$\quad\quad \neg Ga$	
4	$\quad\quad (\neg\forall x \neg Rxx) \lor \neg Ga$	$\lor$I, 3
5	$\quad\quad Ga$	
6	$\quad\quad Rbb$	$\leftrightarrow$E, 2, 5
7	$\quad\quad \exists x Rxx$	$\exists$I, 6
8	$\quad\quad \neg\forall x \neg Rxx$	CQ, 7
9	$\quad\quad (\neg\forall x \neg Rxx) \lor \neg Ga)$	$\lor$I, 8
10	$\quad (\neg\forall x \neg Rxx) \lor \neg Ga)$	TND, 3–4, 5–9
11	$(\neg\forall x \neg Rxx) \lor \neg Ga)$	$\exists$E, 1–10

(ii) $\vdash \forall x((\neg Fx \land Fx) \leftrightarrow (Gx \land \neg Gx))$

1	$\quad \neg Fa \land Fa$	
2	$\quad Fa$	$\land$E, 1
3	$\quad \neg Fa$	$\land$E, 1
4	$\quad \bot$	$\bot$I, 1–3
5	$\quad Ga \land \neg Ga$	$\bot$E, 4
6	$\quad Ga \land \neg Ga$	
7	$\quad Ga$	$\land$E, 6
8	$\quad \neg Ga$	$\land$E, 6
9	$\quad \bot$	$\bot$I, 6–8
10	$\quad \neg Fa \land Fa$	$\bot$E, 9
11	$(\neg Fa \land Fa) \leftrightarrow (Ga \land \neg Ga)$	$\leftrightarrow$I, 1–5, 6–10
12	$\forall x((\neg Fx \land Fx) \leftrightarrow (Gx \land \neg Gx))$	$\forall$I, 11

(iii) $\exists x(Fx \wedge \forall y(Fy \rightarrow y = x)), Fa \vdash Fb \rightarrow a = b$

1	$\exists x(Fx \wedge \forall y(Fy \rightarrow y = x))$	
2	Fa	
3	$\quad Fb \wedge \forall y(Fy \rightarrow y = b)$	
4	$\quad Fb$	$\wedge$E, 3
5	$\quad \forall y(Fy \rightarrow y = b)$	$\wedge$E, 3
6	$\quad Fa \rightarrow a = b$	$\forall$I, 5
7	$\quad a = b$	$\rightarrow$E, 2, 6
8	$\quad \forall y(Fy \rightarrow y = a)$	=E, 5, 7
9	$\forall y(Fy \rightarrow y = a)$	$\exists$E, 1, 3–8
10	$Fb \rightarrow b = a$	
11	$\quad Fb$	
12	$\quad b = a$	$\rightarrow$E, 10–11
13	$\quad a = a$	=E, 12
14	$\quad a = b$	=E, 12–13
15	$Fb \rightarrow a = b$	$\rightarrow$I, 11–14

(iv) $\exists xGx, \exists x(Fx \wedge \forall y(Fy \to y = x)), \forall y(Gy \to Fy) \vdash \exists x\forall y(Gy \leftrightarrow x = y)$

1	$\exists Gx$	
2	$\exists(Fx \wedge \forall y(Fy \to y = x))$	
3	$\forall y(Gy \to Fy)$	
4	$\quad Ga$	
5	$\quad Ga \to Fa$	$\forall$E, 3
6	$\quad Fa$	
7	$\quad\quad Fb \wedge \forall y(Fy \to y = b)$	
8	$\quad\quad Fb$	$\wedge$E, 7
9	$\quad\quad \forall y(Fy \to y = b)$	$\wedge$E, 7
10	$\quad\quad Fa \to a = b$	$\forall$E, 9
11	$\quad\quad a = b$	$\to$E, 6, 10
12	$\quad\quad \forall y(Fy \to y = a)$	=E, 9, 11
13	$\quad \forall y(Fy \to y = a)$	$\exists$E, 2, 7–12
14	$\quad Fb \to b = a$	$\forall$E, 13
15	$\quad\quad Gb$	
16	$\quad\quad Gb \to Fb$	$\forall$E, 3
17	$\quad\quad Fb$	$\to$E, 15–16
18	$\quad\quad b = a$	$\to$E, 14, 17
19	$\quad\quad b = a$	
20	$\quad\quad Gb$	=E, 4, 19
21	$\quad Gb \leftrightarrow a = b$	$\leftrightarrow$I, 15–18, 19–20
22	$\quad \forall y(Gy \leftrightarrow a = y)$	$\forall$I, 21
23	$\quad \exists x\forall y(Gy \leftrightarrow x = y)$	$\exists$I, 22
24	$\exists x\forall y(Gy \leftrightarrow x = y)$	$\exists$E, 1, 4–23

2. Answer all Parts of this question.

(a) Using the following symbolisation key: [60]

Domain: all living creatures
Bx: x is a bulldog
Hx: x is a human
Sx: x skateboards
Axy: x admires y
Bxy: x is better at skateboarding than y
i: Tillman
r: Rodney

Symbolise the following English sentences as best you can in FOL, commenting on any difficulties or limitations you encounter:

(i) Rodney is a human and Tillman is a bulldog, and both can skateboard.

$$(Hr \land Bi) \land (Sr \land Si)$$

(ii) Everybody admires Tillman, because he is a bulldog who can skateboard.

$$\forall x((Si \land Bi) \land Axi)$$

Comment: In the language of FOL, we have no means to capture the 'because' of the original sentence, because all of our connectives are truth–functional.

(iii) Tillman is the skateboarding bulldog.

$$\forall x((Sx \land Bx) \leftrightarrow x = i)$$

(iv) There are skateboarding bulldogs besides Tillman.

$$\exists x \exists y((Sx \land Bx) \land (Sy \land By)) \land ((\neg x = y \land \neg x = i) \land \neg y = i)$$

Comment: More naturally read as 'there is more than one skateboarding bulldog that is not Tillman', but could be read as 'there is more than one skateboarding bulldog adjacent to Tillman', and we haven't been given a relation–symbol for the adjacency relation.

(v) Some human and some bulldog skateboard exactly as well as each other.

$$\exists x \exists y(Hx \land By) \land \neg(Bxy \lor Byx)$$

Comment: It is unclear whether the English should be taken to imply that the equally skilled human and dog can skateboard at all. Here I've taken that to not be so, and that the English would be satisfied by any human and dog neither of whom can skateboard at all.

(vi) Tillman is better at skateboarding than some humans, but he is not better than Rodney.

$$\exists x \exists y(\neg x = y \wedge (Hx \wedge Hy)) \wedge ((Bix \wedge Biy) \wedge \neg Bir)$$

(vii) When it comes to skateboarding, Tillman is the best bulldog.

$$\forall x((Bx \wedge \neg x = i) \to Bix)$$

Comment: Unlike in the English sentence, in FOL we require a clause to express that Tillman is not better at skateboarding than himself, given that nothing in the language rules out 'Bii', the English counterpart of which borders on nonsense.

(viii) No bulldog skateboards better than Tillman does.

$$\neg \exists x(Bx \wedge Bxi)$$

(ix) Tillman admires all skateboarding bulldogs, whether or not they are better at skateboarding than him.

$$\forall x((Bx \wedge Sx) \to Aix)$$

Comment: Any clause in FOL we might add to the effect of 'whether or not they are better at skateboarding' would not have the intended sense; the clause would be a tautology ($Bxi \vee \neg Bxi$), rather than convey the suggestion of Tillman's good character, as the English does.

(x) Two humans who skateboard are better than Rodney at it.

$$\exists x \exists y(((Hx \wedge Hy) \wedge ((\neg x = y \wedge \neg x = r) \wedge \neg y = r)) \wedge (Bxr \wedge Byr))$$

Comment: The English sentence is ambiguous between 'at least two' and 'exactly two', and correspondingly one might formalise (x) either way.

(b) Show that each of the following claims is true: [30]

(i) $\forall x(Fx \leftrightarrow \exists y(\neg x = y \wedge Gy \wedge Rxy)), Fa, Fb, \neg a = b \not\models \exists x \exists y \exists z(\neg x = y \wedge \neg x = z \wedge \neg y = z)$

Domain: $D = \{1, 2\}$
$|F| = D$
$|G| = D$
$|Rxy| = D^2$
$|a| = 1$
$|b| = 2$

In this interpretation, all the sentences to the left of the turnstile are true (since everything is F and bears R to some distinct G in the domain), but the sentence to the right is false, since the domain only has two elements.

(ii) $\exists x \exists y \exists z(\neg x = y \wedge \neg y = z \wedge \neg x = z \wedge (Rxy \leftrightarrow Ryx) \wedge (Rxy \leftrightarrow Rxz)) \not\models \forall x \forall y(Rxy \rightarrow Ryx)$

$D = \{1, 2, 3\}$
$|R| = \{(1, 2), (2, 1), (1, 3)\}$

In this interpretation, the sentence to the left of the turnstile is true, since there are three distinct elements in the domain and letting $x = 1$, $y = 2$, and $z =$, Rxy, Ryx, and Rxz all have the same truth value. However, 1 bears R to 3, but not vice–versa, and hence the sentence to the right of the turnstile is false.

(iii) $\forall x(x = a \vee x = b \vee x = c), \forall x((x = a \vee x = b) \leftrightarrow Fx), \forall x, ((x = b \vee x = c) \leftrightarrow Gx), Rab, \neg Rba \not\models \exists x \exists y, \exists z(\neg x = y \wedge \neg x = z \wedge \neg y = z)$

$D = \{0, 1\}$
$|F| = \{0\}$
$|G| = D$
$|R| = \{(0, 1)\}$
$|a| = |b| = 0$
$|c| = 1$

In this interpretation, everything is a, b, or c. Everything is a or b iff it is F, and everything is b or c, and G. Rab and Rba, hence all the sentences to the left of the turnstile are true, and the sentence to the right is false, since the domain contains only two elements.

(iv) $\exists x \neg \exists y Ryx, \forall x \exists y Rxy, \forall x \forall y \forall z((Rxy \wedge Ryz) \rightarrow Rxz) \not\models \exists x \forall y \forall z((Rxy \wedge Rxz \rightarrow y = z)$

$D = \mathbb{N}$ (the natural numbers)
$|Rxy| = \{(x, y) : x < y\}$

In this interpretation, all of the sentences to the left of the turnstile are true: 0 is least in the natural numbers (i.e. nothing is smaller than it); every number is smaller than some other number; and the $<$ relation is transitive over $\mathbb{N}$. However, no number is smaller than exactly one number, so the sentence to the right of the turnstile is false.

(c) Explain why we insist that the domain of quantification for any interpretation in FOL must always contain at least one object. [10]

There are two primary motivations for this insistence; one proof theoretic, and one semantic. Our natural deduction system licences the following proof:

1	$a = a$	=I
2	$\exists x x = x$	$\exists$I, 1

So, if our domain were allowed to be empty, our proof system would be unsound, since $\exists x x = x$ is true iff there is some self–identical object in the domain. Our second reason is semantic. If nothing in the domain is F, then $\forall x(Fx \rightarrow ...)$ is vacuously true. So if we allowed an empty domain, we would lose a good deal of our logical truths, for example $\forall x \neg(x = x \rightarrow \neg x = x)$ would no longer be logically true because any interpretation with an empty domain would validate $\forall x(x = x \rightarrow \neg x = x)$; it would be vacuously true because the domain contains no self–identical object. In other words, we insist on a non-empty domain because to do otherwise would require a wholesale revision of our logic, both semantics and deductive system.

3. (a) Define the following notions from set theory: Cartesian Product, Empty Set, Power Set, Proper Subset. [10]

$A \times B = \{(a, b) | a \in A \wedge b \in b\}$
$\emptyset = \{x | \neg x = x\}$
$\mathcal{P}(A) = \{x | x \subseteq A\}$
$A \subset B \leftrightarrow (A \subseteq B \wedge \neg A = B)$

(b) Give examples of the following: [40]

i. A set with exactly one subset. $\emptyset$
ii. A set with exactly one proper subset. $\{\emptyset\}$
iii. A set of which at least one member is also a subset. $\{\emptyset\}$
iv. Two sets whose Cartesian product has exactly six members. $A = \{0, 1\}, B =$

$\{2, 3, 4\}$
v. A set whose power set has the set containing the empty set as a member. $\{\emptyset\}$
vi. A set whose intersection with its own power set is non-empty. $\{\emptyset\}$
vii. A finite, non-empty set whose members include all members of its own members. $\{\emptyset\}$
viii. A set whose members are all sets. $\{\emptyset\}$

(c) Define the following notions from the theory of relations: Ancestral, Converse, Equivalence relation, equivalence class. [20]

$$R^*xy = \{(x, y) | \exists z_1 ... \exists z_n (Rxz_1 \wedge Rz_1 z_2 \wedge ... \wedge Rz_n y\}$$

$$R^{'}xy = \{(x, y) | Ryx\}$$

A relation is an equivalence relation over a given domain if and only if it is relfexive, symmetric and transitive over the domain, i.e. $\forall x Rxx$, $\forall x \forall y (Rxy \rightarrow Ryx)$, and $\forall x \forall y \forall z ((Rxy \wedge Ryz) \rightarrow Rxz)$.

An equivalence class is a member of a partition of a set given by an equivalence relation. If $a \in X$ and R is an equivalence relation, then $[a] = \{x \in X | Rax\}$. Hence, for any $a, b \in X$ $[a] = [b]$ or $[a] \cap [b] = \emptyset$.

(d) Each of the following statements is false. In each case give a counterexample, being careful to specify a domain for each answer. [30]

In all cases, let $D = \mathbb{N}$.

i. Every relation is its own ancestral. Let R be the predecessor relation (i.e. Rxy if and only if $x + 1 = y$). The ancestral R^* is the relation $<$.

ii. Every relation is its own converse. Let R be the relation x divides y. This relation includes every pair $(n, 0)$ for all $\neg n = 0$, but $R^{'}$ includes no such pair, since 0 doesn't divide any number at all.

iii. Every reflexive and symmetric relation is transitive. Let $R = \{(x, y) | x = y \vee x = y + 1 \vee x + 1 = y\}$. This is reflexive since every number is self-identical, and symmetric, since if $x = y + 1$ then $y + 1 = x$ and vice–versa. But it isn't transitive, since $R01$ and $R12$ and $\neg R02$.

iv. Every symmetric and transitive relation is reflexive. Let R be the empty relation.

v. Every reflexive and transitive relation is symmetric. Let R be $\leq$.

4. (a) Write down the axioms of the probability calculus. Define Conditional probability. State any form of Bayes's Theorem. [15]

Probability Axioms:

$Pr(V) = 1$, where V is our sample space (set of possible outcomes).
$Pr(X) \geq 0$ where $X \in \mathcal{P}(V)$
if $X \cap Y = \emptyset$ then $Pr(X \cup Y) = Pr(X) + Pr(Y)$

$Pr(A|B) = \frac{Pr(A \cap B)}{Pr(B)}$

Bayes's Theorem: $Pr(H|E) = \frac{(Pr(E|H)Pr(H))}{((PrE|H)Pr(H)) + (Pr(E|\neg H)Pr(\neg H))}$

(b) Condition C occurs in 0.01% of the population. Suppose that a test C is developed: It is positive on 95% of occasions when C is absent and on 1% of occasions when C is absent. You take the test and it is positive. What is the probability that you have C? [25]

Letting 'C' stand for 'You have C' and 'P' for 'You test positive for C', Bayes's theorem tells us that:

$Pr(C|P) = \frac{Pr(P|C)Pr(C)}{(Pr(P|C)Pr(C)) + (Pr(P|\neg C)Pr(\neg C))}$

So, $Pr(C|P) = \frac{(0.95 \times 0.0001)}{(0.95 \times 0.0001) + (0.01 \times 0.9999)}$

i.e., $Pr(C|P) = \frac{0.000095}{0.000095 + 0.009999}$

Hence, $Pr(C|P) = \frac{0.000095}{0.010094}$

Giving us $Pr(C|P) \approx 0.0094$

(c) I draw two cards from a standard pack without replacement. Calculate the following probabilities [60]:

Let 'H_n' be 'the n^{th} drawn card is a heart', 'Q_n^H' be 'the n^{th} card drawn is the Queen of Hearts' and 'A_n' be 'the n^{th} card drawn is an ace'.

i. The probability that the second is a heart given that the first is.

We have it that $Pr(H_2|H_1) = \frac{Pr(H_2 \cap H_1)}{Pr(H_1)}$. We also know that $Pr(H_1) = \frac{1}{4}$, given that the pack is standard. There are 13 hearts in the pack, and we are drawing without replacement, so $Pr(H_2 \cap H_1) = \frac{13 \times 12}{52 \times 51} = \frac{156}{2652}$. Dividing by $\frac{1}{4}$ for the answer gives us $Pr(H_2|H_1) = \frac{624}{2652} \approx 0.24$.

ii. The probability that the second is a heart given that the first is the queen of hearts.

$Pr(H_2|Q_1^H) = \frac{Pr(H_2 \cap Q_1^H)}{Pr(Q_1^H)}$. We know that $Pr(Q_1^H) = \frac{1}{52}$, and also that the number of possible draws is 52×51. So, $Pr(H_2 \cap Q_1^H) = \frac{12}{2652}$, given that there are 12 ordered pairs in our event space such that the first is the queen of hears and the second is any other heart. So we divide by $\frac{1}{52}$ to get our answer: $Pr(H_2|Q_1^H) = \frac{624}{2652} \approx 0.24$.

iii. The probability that at least one of them is an ace.

The pack contains 48 non-aces, and 4 aces. So we have 4×48 pairs where the first is an ace and the second isn't, the same number of pairs where the first isn't and the second is, and 12 pairs where both cards are aces. There are 51×52 outcomes, so $Pr(A_1 \vee A_2) = \frac{(4\times48)+(4\times48)+12)}{52\times51} = \frac{396}{2652} \approx 0.15$.

iv. The probability that the second is at least as high as the first in the following ranking: 2–10, J, Q, J, A.

Rank the cards from 1 through 13. For sequences beginning with a card of rank n, there are three pairs where the rank of the second member is n (i.e. where the second is the same picture for a different suit), and $4 \times (13 - n)$ pairs where the second is greater than the first. Since there are four cards of any given rank, the number of pairs satisfying the hypothesis (i.e. the rank of the second is greater than or equal to the rank of the first) is given by: $\sum_{n=1}^{13}(4 \times (3 + (4 \times (13 - n))))$.
Going through the cases, this gives us $204 + 188 + 172 + 156 + 140 + 124 + 108 + 92 + 76 + 60 + 44 + 28 + 12 = 1404$. Hence the probability that the second card is of equal to or greater rank than the first is $\frac{1404}{2652} \approx 0.53$